Worksheet - Roots of polynomials, series and Maclaurin

Year: Further Maths · 20

Show your working.

Question 1

Question 1 – Sums and products of roots *(Section A · Skills practice)*

Answer each part without solving the equations.

a. The roots of $2x^2 - 5x + 1 = 0$ are α and β. Find $\alpha + \beta$ and $\alpha\beta$.
b. Hence find $\alpha^2 + \beta^2$ and $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$.
c. The roots of $x^3 + 2x^2 - 3x + 5 = 0$ are α, β and γ. Write down $\sum \alpha$, $\sum \alpha\beta$ and $\alpha\beta\gamma$.
d. Find $\alpha^2 + \beta^2 + \gamma^2$ for the cubic in part (c).

Question 2

Question 2 – Equations with transformed roots *(Section A · Skills practice)*

The cubic equation $x^3 - 3x^2 + 5x - 2 = 0$ has roots α, β and γ. Use a substitution to find a cubic equation, with integer coefficients, whose roots are:

a. 2α, 2β, 2γ
b. $\alpha + 1$, $\beta + 1$, $\gamma + 1$
c. $\dfrac{1}{\alpha}$, $\dfrac{1}{\beta}$, $\dfrac{1}{\gamma}$
d. Check your answer to part (a) using the sum of the roots.

Question 3

Question 3 – Roots of a quartic *(Section A · Skills practice)*

The quartic equation $x^4 - 2x^3 + 3x^2 - x + 4 = 0$ has roots α, β, γ and δ.

a. Write down $\sum \alpha$, $\sum \alpha\beta$, $\sum \alpha\beta\gamma$ and $\alpha\beta\gamma\delta$.
b. Find $\sum \alpha^2$.
c. Explain why the equation cannot have four real roots.
d. Find $\sum \dfrac{1}{\alpha}$.

Question 4

Question 4 – Using the standard sums *(Section A · Skills practice)*

Use $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$, $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$ and $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$.

a. Show that $\sum_{r=1}^{n}(6r^2+2r) = 2n(n+1)^2$.

b. Hence find $\sum_{r=1}^{20}(6r^2+2r)$.

c. Find $\sum_{r=11}^{20} r^3$.

d. Check the formula in part (a) for $n = 2$.

Question 5

Question 5 – The method of differences *(Section A · Skills practice)*

This question is about $S_n = \sum_{r=1}^{n} \frac{1}{r(r+2)}$.

a. Express $\frac{1}{r(r+2)}$ in partial fractions.

b. Write out the first three and the last two terms of the sum using your partial fractions, and show which terms cancel.

c. Show that $S_n = \frac{3}{4} - \frac{2n+3}{2(n+1)(n+2)}$.

d. Find the sum to infinity.

Question 6

Question 6 – Maclaurin series from the definition *(Section A · Skills practice)*

The Maclaurin series of $f(x)$ is $f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$

a. Find the series for e^{2x} up to the term in x^3.

b. By differentiating repeatedly, find the series for $\ln(1+x)$ up to the term in x^4, and state the values of x for which it is valid.

c. Use the standard series for $\cos x$ to write down the first three non-zero terms of the series for $\cos 3x$.

Question 7

Question 7 – The dimensions of a box *(Section B · Real-life practice)*

A company designs a cuboid box whose length, width and height in decimetres are the three roots α, β, γ of $x^3 - 12x^2 + 44x - 48 = 0$.

a. Find the total length of the 12 edges of the box.
b. Find the total surface area and the volume of the box.
c. Find the length of the longest rod that fits inside the box (the space diagonal), in exact form.
d. Show that $x = 2$ is a root, find the other two dimensions and check your answers.

Question 8

Question 8 – A display of boxes *(Section B · Real-life practice)*

A shop builds a square-based display of boxes. The top layer is 1 box, the next is a 2×2 square, and layer r is an $r \times r$ square.

Layers	1	2	3	4	5
Boxes in bottom layer	1	4	9		
Total boxes	1	5	14		

a. Complete the table.
b. Write the total number of boxes in an n-layer display as a sum, and find a formula for it.
c. The shop has 1000 boxes. Find the number of layers in the largest complete display.
d. Each box weighs 0.6 kg. Find the weight carried by the floor for a 10-layer display.

Question 9

Question 9 – A pendulum clock *(Section B · Real-life practice)*

In the theory of a pendulum clock, $\sin\theta$ is often replaced by θ for small angles in radians.

a. Write down the Maclaurin series for $\sin\theta$ up to the term in θ^5.
b. The pendulum swings to $\theta = 0.2$ rad. Find the percentage error in the approximation $\sin\theta \approx \theta$.
c. Find the value given by $\sin\theta \approx \theta - \frac{\theta^3}{6}$ at $\theta = 0.2$ and comment on its accuracy.
d. Use the series to find $\displaystyle\lim_{\theta \to 0} \frac{\theta - \sin\theta}{\theta^3}$.

Question 10

Question 10 – Inside a calculator *(Section B · Real-life practice)*

Early calculators worked out values of functions using a few terms of a series. A programmer tests two series.

Value	Series	Terms used
$e^{0.1}$	$1 + x + \frac{x^2}{2} + \frac{x^3}{6}$	4
$\ln 1.2$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$	4

a. Find the approximation to $e^{0.1}$ and compare it with the calculator value 1.105171.
b. Find the approximation to $\ln 1.2$ and compare it with 0.182322.
c. Explain why the series for $e^{0.1}$ is more accurate than the series for $\ln 1.2$ with the same number of terms.
d. Explain why the series for $\ln(1 + x)$ cannot be used directly to find $\ln 3$.

Question 11

Question 11 – The sag of a power cable *(Section B · Real-life practice)*

A power cable between two pylons 20 m apart hangs in the curve $y = 20\cosh\frac{x}{20}$, with the lowest point at $x = 0$ and the pylons at $x = \pm 10$.

a. Write down the Maclaurin series for $\cosh u$ up to the term in u^4.
b. The sag is the height of the supports above the lowest point. Show that the sag is $20\left(\cosh\frac{1}{2} - 1\right)$.
c. Use the first two non-constant terms of the series to estimate the sag, to 4 decimal places.
d. Compare with the exact value, and explain why the simple parabola $y = 20 + \frac{x^2}{40}$ is a good model of the cable.

Question 12

Question 12 – Three ages *(Section B · Real-life practice)*

The ages of three cousins are the roots α, β, γ of the equation $x^3 - 39x^2 + 491x - 1989 = 0$.

a. Find the sum of their ages and the product of their ages.
b. Find $\alpha^2 + \beta^2 + \gamma^2$.
c. Find a cubic equation whose roots are their ages in 5 years' time.
d. Given that the youngest cousin is 9, find the other two ages and check part (b).

Question 13

Question 13 – A pattern of square tiles *(Section B · Real-life practice)*

An artist makes a mosaic from square tiles. Tile r has side $(2r-1)$ cm, so the tiles have sides 1 cm, 3 cm, 5 cm and so on.

a. Show that the total area of the first n tiles is $\sum_{r=1}^{n}(4r^2-4r+1)$.
b. Use the standard sums to show that this equals $\frac{1}{3}n(2n-1)(2n+1)$.
c. Find the total area of the first 20 tiles.
d. The artist has a wall of area 1 m². Find the largest number of tiles that fit, ignoring gaps.

Question 14

Question 14 – Checking a diagonal *(Section B · Real-life practice)*

A builder checks that a frame is square by measuring a diagonal. One side is 2 m and the other side is 0.4 m.

a. Show that the diagonal is $2\sqrt{1.04}$ m.
b. Find the Maclaurin series of $\sqrt{1+x}$ up to the term in x^2.
c. Use the series with $x=0.04$ to estimate the diagonal to 4 decimal places, without a calculator.
d. Find the error of the estimate in millimetres.

Question 15

Question 15 – Powers of the roots *(Section C · Problem solving)*

The equation $x^3-7x+6=0$ has roots α, β, γ.

Quantity	$\sum\alpha$	$\sum\alpha\beta$	$\alpha\beta\gamma$	$\sum\alpha^2$	$\sum\alpha^3$
Value					

a. Fill in the first three entries of the table using the coefficients.
b. Find $\sum\alpha^2$.
c. Explain why $\alpha^3=7\alpha-6$, and hence find $\sum\alpha^3$.
d. By substituting $x=\sqrt{w}$, find a cubic equation whose roots are α^2, β^2, γ^2.
e. Solve the original equation and check all your answers.

Question 16

Question 16 – A quartic with complex roots *(Section C · Problem solving)*

The quartic $f(x)=x^4-4x^3+6x^2-4x-15$ has real coefficients, and $1+2i$ is a root

of $f(x) = 0$.

a. Write down another complex root and find the quadratic factor with these two roots.
b. Find the other quadratic factor and hence all four roots.
c. Verify that the sum and the product of the four roots agree with the coefficients of f.
d. Find $\sum \alpha^2$ from the coefficients and check it using the roots.
e. Show that $f(x) = (x-1)^4 - 16$ and explain how this gives the roots more quickly.

Question 17

Question 17 – Two telescoping sums *(Section C · Problem solving)*

A spreadsheet shows partial sums of two series.

n	1	2	3	4
$\sum_{r=1}^{n} \frac{2}{(2r-1)(2r+1)}$	$\frac{2}{3}$	$\frac{4}{5}$		
$\sum_{r=1}^{n} r \cdot r!$	1	5		

a. Complete the table and suggest a formula for each sum.
b. Show that $\frac{2}{(2r-1)(2r+1)} = \frac{1}{2r-1} - \frac{1}{2r+1}$, and use the method of differences to prove your first formula.
c. Show that $r \cdot r! = (r+1)! - r!$, and prove your second formula.
d. Find the smallest n for which the first sum is greater than 0.999.
e. Explain the difference between the behaviour of the two sums as $n \to \infty$.

Question 18

Question 18 – Series of products and composites *(Section C · Problem solving)*

Use the standard Maclaurin series for e^x, $\sin x$, $\cos x$ and $\ln(1+u)$.

a. Find the series for $e^x \sin x$ up to the term in x^3.
b. Hence find $\lim_{x \to 0} \frac{e^x \sin x - x - x^2}{x^3}$.
c. Write $\cos x = 1 + u$, where $u = -\frac{x^2}{2} + \frac{x^4}{24} - \cdots$, and show that $\ln(\cos x) = -\frac{x^2}{2} - \frac{x^4}{12} + \cdots$
d. Use part (c) to estimate $\ln(\cos 0.1)$ and compare with a calculator.
e. Differentiate the series in part (c) to obtain the series for $\tan x$ up to the term in x^3.

Question 19

Question 19 – Areas under a bell curve *(Section C · Problem solving)*

Statisticians need values of $\int_0^a e^{-x^2}\,dx$, which cannot be found with elementary functions. The table shows the first terms of the integral for $a = 0.5$.

Term	1st	2nd	3rd	4th
Value	0.5			

a. Use the series for e^u to show that $e^{-x^2} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots$

b. Integrate term by term to find a series for $\int_0^a e^{-x^2}\,dx$.

c. Complete the table for $a = 0.5$ and estimate the integral to 5 decimal places.

d. The exact value is 0.461281 to 6 decimal places. Find the error, and explain how the size of the next term predicts it.

e. The probability that a normal variable lies within $\frac{1}{\sqrt{2}}$ standard deviations of the mean is $\frac{2}{\sqrt{\pi}}\int_0^{0.5} e^{-x^2}\,dx \times \ldots$ Explain why series are used to build statistical tables.

Question 20

Question 20 – Challenge: a series for pi *(Section D · Challenge)*

This question shows how series were used to calculate π by hand.

a. Write down the series for $\frac{1}{1+t^2}$ as a geometric series, and integrate from 0 to x to show that $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

b. Put $x = 1$ to obtain a series for π, and find the estimate given by its first four terms.

c. Put $x = \frac{1}{\sqrt{3}}$ and show that $\pi = 2\sqrt{3}\left(1 - \frac{1}{3\times 3} + \frac{1}{5\times 9} - \frac{1}{7\times 27} + \cdots\right)$.

d. Find the estimate given by the four terms in part (c), to 5 decimal places.

e. Explain why the series in part (c) is much better than the one in part (b).