# Answers - Roots of polynomials, series and Maclaurin

Year: Further Maths

## Answer to Question 1

Method: for $ax^2+bx+c=0$, $\alpha+\beta=-\dfrac{b}{a}$ and $\alpha\beta=\dfrac{c}{a}$; for a cubic $ax^3+bx^2+cx+d=0$, $\sum\alpha=-\dfrac{b}{a}$, $\sum\alpha\beta=\dfrac{c}{a}$, $\alpha\beta\gamma=-\dfrac{d}{a}$. Other symmetric expressions are rewritten in terms of these, for example $\sum\alpha^2=\left(\sum\alpha\right)^2-2\sum\alpha\beta$.

a. Here $a=2$, $b=-5$, $c=1$. For $ax^2+bx+c=0$: $\alpha+\beta=-\dfrac{b}{a}=\dfrac{5}{2}$ and $\alpha\beta=\dfrac{c}{a}=\dfrac{1}{2}$.

b. Express each quantity in terms of $\alpha+\beta$ and $\alpha\beta$.
$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\dfrac{25}{4}-1=\dfrac{21}{4}$. $\dfrac{1}{\alpha}+\dfrac{1}{\beta}=\dfrac{\alpha+\beta}{\alpha\beta}=\dfrac{5/2}{1/2}=\mathbf{5}$.

c. Here $a=1$, $b=2$, $c=-3$, $d=5$. $\sum\alpha=-\dfrac{2}{1}=-2$, $\sum\alpha\beta=-3$, $\alpha\beta\gamma=-\dfrac{5}{1}=-5$.

d. Use the identity $(\alpha+\beta+\gamma)^2=\sum\alpha^2+2\sum\alpha\beta$.
$\sum\alpha^2=\left(\sum\alpha\right)^2-2\sum\alpha\beta=4-2(-3)=\mathbf{10}$.

## Answer to Question 2

a. Let $w=2x$, so $x=\frac{w}{2}$: $\frac{w^3}{8}-\frac{3w^2}{4}+\frac{5w}{2}-2=0$. Multiply by 8:
$w^3-6w^2+20w-16=0$.

b. Let $w=x+1$, so $x=w-1$:
$(w-1)^3-3(w-1)^2+5(w-1)-2=w^3-3w^2+3w-1-3w^2+6w-3+5w-5-2$, so $w^3-6w^2+14w-11=0$.

c. Let $w=\frac{1}{x}$, so $x=\frac{1}{w}$: $\frac{1}{w^3}-\frac{3}{w^2}+\frac{5}{w}-2=0$. Multiply by $w^3$:
$1-3w+5w^2-2w^3=0$, i.e. $2w^3-5w^2+3w-1=0$.

d. The original sum of roots is 3, so the new sum should be $2\times3=6$, and the equation in (a) gives $-\frac{-6}{1}=6$ ✓

## Answer to Question 3

Method: for $ax^2+bx+c=0$, $\alpha+\beta=-\dfrac{b}{a}$ and $\alpha\beta=\dfrac{c}{a}$; for a quartic the pattern

continues with alternating signs; for a cubic $ax^3+bx^2+cx+d=0$, $\sum\alpha=-\dfrac{b}{a}$, $\sum\alpha\beta=\dfrac{c}{a}$, $\alpha\beta\gamma=-\dfrac{d}{a}$. Other symmetric expressions are rewritten in terms of these, for example $\sum\alpha^2=\left(\sum\alpha\right)^2-2\sum\alpha\beta$.

a. $\sum\alpha=$ **2**, $\sum\alpha\beta=$ **3**, $\sum\alpha\beta\gamma=-(-1)=$ **1**, $\alpha\beta\gamma\delta=$ **4**.
b. $\sum\alpha^2=\left(\sum\alpha\right)^2-2\sum\alpha\beta=4-6=-2$
c. If all four roots were real, their squares would be non-negative and $\sum\alpha^2\geq 0$. Since $\sum\alpha^2=-2$, **at least two roots are not real**.
d. $\sum\dfrac{1}{\alpha}=\dfrac{\sum\alpha\beta\gamma}{\alpha\beta\gamma\delta}=\dfrac{1}{4}$

## Answer to Question 4

Method: split the sum into multiples of $\sum r^3$, $\sum r^2$, $\sum r$ and $\sum 1=n$, substitute the standard formulae, and factorise out common factors such as $n(n+1)$ as early as possible.

a. $6\times\frac{1}{6}n(n+1)(2n+1)+2\times\frac{1}{2}n(n+1)=n(n+1)(2n+1)+n(n+1)=n(n+1)(2n+2)=2n(n+1)^2$ ✓ Factorising $n(n+1)$ first keeps the algebra short.
b. Substitute $n=20$ into the formula from part a). $2\times 20\times 21^2=$ **17 640**
c. A sum that starts at $r=11$ is the sum to 20 minus the sum to 10.
$\displaystyle\sum_{r=1}^{20}r^3-\sum_{r=1}^{10}r^3=\tfrac{1}{4}(400)(441)-\tfrac{1}{4}(100)(121)=44\,100-3025=$ **41 075**
d. Add the first two terms directly and compare with the formula. $n=2$: $(6+2)+(24+4)=36$ and $2\times 2\times 9=36$ ✓

## Answer to Question 5

a. $\dfrac{1}{r(r+2)}=\dfrac{1}{2}\left(\dfrac{1}{r}-\dfrac{1}{r+2}\right)$
b. $\frac{1}{2}\left[\left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{n-1}-\frac{1}{n+1}\right)+\left(\frac{1}{n}-\frac{1}{n+2}\right)\right]$. Every negative term cancels with a positive term two places later, leaving $1$**,** $\frac{1}{2}$**,** $-\frac{1}{n+1}$ **and** $-\frac{1}{n+2}$.
c. $S_n=\frac{1}{2}\left(\frac{3}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right)=\frac{3}{4}-\frac{1}{2}\cdot\dfrac{(n+2)+(n+1)}{(n+1)(n+2)}=\dfrac{3}{4}-\dfrac{2n+3}{2(n+1)(n+2)}$ ✓
d. As $n\to\infty$ the fraction tends to 0: $\dfrac{3}{4}$.

## Answer to Question 6

a. $f=e^{2x}$, $f'=2e^{2x}$, $f''=4e^{2x}$, $f'''=8e^{2x}$; at 0 these are $1,2,4,8$. So

$e^{2x} = 1 + 2x + \frac{4}{2}x^2 + \frac{8}{6}x^3 + \cdots = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \cdots$

b. $f = \ln(1+x)$: $f(0) = 0$, $f' = (1+x)^{-1} \to 1$, $f'' = -(1+x)^{-2} \to -1$, $f''' = 2(1+x)^{-3} \to 2$, $f^{(4)} = -6(1+x)^{-4} \to -6$. So $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$, valid for $-1$.

c. $\cos u = 1 - \frac{u^2}{2} + \frac{u^4}{24} - \cdots$ with $u = 3x$: $1 - \frac{9}{2}x^2 + \frac{27}{8}x^4 - \cdots$

## Answer to Question 7

a. Each dimension appears 4 times: $4(\alpha + \beta + \gamma) = 4 \times 12 = \mathbf{48\ dm}$.

b. Surface area $= 2(\alpha\beta + \beta\gamma + \gamma\alpha) = 2 \times 44 = \mathbf{88\ dm^2}$. Volume $= \alpha\beta\gamma = \mathbf{48\ dm^3}$.

c. $d^2 = \alpha^2 + \beta^2 + \gamma^2 = 12^2 - 2 \times 44 = 56$, so $d = \sqrt{56} = 2\sqrt{14} \approx 7.48$ **dm**.

d. $8 - 48 + 88 - 48 = 0$ ✓. Dividing: $x^3 - 12x^2 + 44x - 48 = (x-2)(x^2 - 10x + 24) = (x-2)(x-4)(x-6)$: **2, 4 and 6 dm**. Edges $4 \times 12 = 48$ ✓, area $2(8 + 24 + 12) = 88$ ✓, volume 48 ✓, diagonal $\sqrt{4 + 16 + 36} = \sqrt{56}$ ✓.

## Answer to Question 8

Method: split the sum into multiples of $\sum r^3$, $\sum r^2$, $\sum r$ and $\sum 1 = n$, substitute the standard formulae, and factorise out common factors such as $n(n+1)$ as early as possible.

a. Layer $r$ has $r^2$ boxes; add each new layer to the previous total. Bottom layers 16, 25; totals 30, 55. So 4 layers need 30 boxes and 5 layers need 55 boxes.

b. $\displaystyle\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$ Check: $n = 3$ gives $\frac{1}{6} \times 3 \times 4 \times 7 = 14$.

c. Try values of $n$ near the answer and compare the totals with 1000. 13 layers need $\frac{1}{6} \times 13 \times 14 \times 27 = 819$ boxes; 14 layers need $1015 > 1000$. So **13 layers** (819 boxes).

d. Find the number of boxes, then multiply by the mass of each box. $\frac{1}{6} \times 10 \times 11 \times 21 = 385$ boxes; $385 \times 0.6 = \mathbf{231\ kg}$.

## Answer to Question 9

a. $\sin\theta = \theta - \dfrac{\theta^3}{6} + \dfrac{\theta^5}{120} - \cdots$

b. $\sin 0.2 = 0.198669$. Error $= \dfrac{0.2 - 0.198669}{0.198669} \times 100 = \mathbf{0.67\%}$.

c. $0.2 - \dfrac{0.008}{6} = 0.2 - 0.001333 = \mathbf{0.198667}$, which agrees with $\sin 0.2$ to **5 decimal places** (error about $3 \times 10^{-6}$, the size of the next term $\frac{0.2^5}{120}$).

d. $\theta - \sin\theta = \dfrac{\theta^3}{6} - \dfrac{\theta^5}{120} + \cdots$, so the quotient is $\dfrac{1}{6} - \dfrac{\theta^2}{120} + \cdots \to \dfrac{1}{6}$.

## Answer to Question 10

a. $1 + 0.1 + 0.005 + 0.000167 =$ **1.105167**, error about $4 \times 10^{-6}$.
b. $0.2 - 0.02 + 0.002667 - 0.0004 =$ **0.182267**, error about $5.5 \times 10^{-5}$.
c. The terms of the exponential series are divided by factorials, so they become small much faster; in the logarithm series the terms only shrink like $\frac{x^n}{n}$, so **more terms are needed**.
d. $\ln 3$ needs $x = 2$, outside the interval of validity $-1$: **the series diverges**. (One can use $\ln 3 = -\ln\frac{1}{3}$ with $x = -\frac{2}{3}$ instead.)

## Answer to Question 11

a. $\cosh u = 1 + \dfrac{u^2}{2} + \dfrac{u^4}{24} + \cdots$
b. At $x = \pm 10$, $y = 20\cosh\frac{1}{2}$; at $x = 0$, $y = 20$. **Sag** $= 20\cosh\frac{1}{2} - 20 = 20\left(\cosh\frac{1}{2} - 1\right)$ ✓
c. $20\left(\frac{0.25}{2} + \frac{0.0625}{24}\right) = 20(0.125 + 0.0026042) =$ **2.5521 m**
d. Exact: $20(1.1276260 - 1) = 2.5525$ m, so the estimate is within 0.4 mm. The parabola keeps only the $u^2$ term, giving 2.5 m: **the higher terms are very small** for a shallow cable.

## Answer to Question 12

a. Sum **39 years**; product **1989**.
b. $39^2 - 2 \times 491 = 1521 - 982 =$ **539**
c. Let $w = x + 5$, so $x = w - 5$. Substituting and expanding: $w^3 - 54w^2 + 956w - 5544 = 0$. (Check: the new sum $54 = 39 + 15$ ✓.)
d. Divide by $(x - 9)$: $x^2 - 30x + 221 = (x - 13)(x - 17)$: **13 and 17**. $81 + 169 + 289 = 539$ ✓

## Answer to Question 13

a. Tile $r$ has area $(2r - 1)^2 = 4r^2 - 4r + 1$; adding for $r = 1$ to $n$ gives the sum ✓
b. $4 \times \frac{1}{6}n(n+1)(2n+1) - 4 \times \frac{1}{2}n(n+1) + n = \frac{1}{3}n\left(2(n+1)(2n+1) - 6(n+1) + 3\right) = \frac{1}{3}n(4n^2 - 1) = \frac{1}{3}n(2n-1)(2n+1)$ ✓
c. $\frac{1}{3} \times 20 \times 39 \times 41 =$ **10 660 cm²**
d. $1$ m² $= 10\,000$ cm². 19 tiles: $\frac{1}{3} \times 19 \times 37 \times 39 = 9139$ cm²; 20 tiles exceed $10\,000$.

**19 tiles**.

## Answer to Question 14

Method: the Maclaurin series is $f(0) + f'(0)x + \dfrac{f''(0)}{2!}x^2 + \cdots$. Often it is quicker to substitute into a known series (for $e^x$, $\sin x$, $\cos x$, $\ln(1+x)$, $(1+x)^n$) and to keep the range of validity of that series.

a. Use Pythagoras and take out the factor 4 from under the root.
$d = \sqrt{4 + 0.16} = \sqrt{4(1 + 0.04)} = 2\sqrt{1.04}$ ✓

b. Differentiate $(1+x)^{1/2}$ twice, or use the binomial series with $n = \frac{1}{2}$.
$f = (1+x)^{1/2}$, $f'(0) = \frac{1}{2}$, $f''(0) = -\frac{1}{4}$: $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots$

c. Substitute $x = 0.04$: $\frac{1}{2}x = 0.02$ and $\frac{1}{8}x^2 = 0.0002$.
$2\left(1 + 0.02 - \frac{1}{8}(0.0016)\right) = 2(1.0198) =$ **2.0396 m**

d. Exact: $\sqrt{4.16} = 2.039608$ m, so the error is about $0.000008$ m = **0.008 mm**, negligible for a builder.

## Answer to Question 15

a. $\sum \alpha =$ **0**, $\sum \alpha\beta = -7$, $\alpha\beta\gamma = -6$

b. $0^2 - 2(-7) =$ **14**

c. Each root satisfies the equation, so $\alpha^3 - 7\alpha + 6 = 0$. Adding for all three roots: $\sum \alpha^3 = 7\sum \alpha - 18 = -18$.

d. $x^3 - 7x = -6$ gives $x(x^2 - 7) = -6$. Square: $x^2(x^2-7)^2 = 36$, so $w(w-7)^2 = 36$: $w^3 - 14w^2 + 49w - 36 = 0$.

e. $x = 1$ is a root; $x^3 - 7x + 6 = (x-1)(x^2 + x - 6) = (x-1)(x-2)(x+3)$: roots $1, 2, -3$. Checks: $\sum \alpha^2 = 1 + 4 + 9 = 14$ ✓, $\sum \alpha^3 = 1 + 8 - 27 = -18$ ✓, and $w$-roots $1, 4, 9$ satisfy $(w-1)(w-4)(w-9) = w^3 - 14w^2 + 49w - 36$ ✓

## Answer to Question 16

a. The conjugate $1 - 2i$ is a root. $(x - 1 - 2i)(x - 1 + 2i) = (x-1)^2 + 4 = x^2 - 2x + 5$.

b. Dividing, $f(x) = (x^2 - 2x + 5)(x^2 - 2x - 3) = (x^2 - 2x + 5)(x-3)(x+1)$. Roots $1 \pm 2i$, $3$, $-1$.

c. Sum $= (1 + 2i) + (1 - 2i) + 3 - 1 = 4 = -\frac{-4}{1}$ ✓. Product $= (1+4)(3)(-1) = -15$ ✓.

d. $\sum \alpha^2 = 4^2 - 2 \times 6 = 4$. From the roots:
$(1 + 2i)^2 + (1 - 2i)^2 + 9 + 1 = (-3 + 4i) + (-3 - 4i) + 10 = 4$ ✓

e. $(x-1)^4 = x^4 - 4x^3 + 6x^2 - 4x + 1$, so $f(x) = (x-1)^4 - 16$ ✓. Then $(x-1)^4 = 16$ gives $x - 1 = 2, -2, 2i, -2i$: $x = 3, -1, 1 + 2i, 1 - 2i$.

## Answer to Question 17

a. First row: $\frac{6}{7}$, $\frac{8}{9}$; suggest $\dfrac{2n}{2n+1}$. Second row: $5+18=23$, $23+96=119$; suggest $(n+1)!-1$.

b. $\dfrac{1}{2r-1}-\dfrac{1}{2r+1}=\dfrac{(2r+1)-(2r-1)}{(2r-1)(2r+1)}=\dfrac{2}{(2r-1)(2r+1)}$ ✓. Sum:
$\left(1-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)=1-\dfrac{1}{2n+1}=\dfrac{2n}{2n+1}$ ✓

c. $(r+1)!-r!=r!\,(r+1-1)=r\cdot r!$ ✓. Sum:
$(2!-1!)+(3!-2!)+\cdots+((n+1)!-n!)=(n+1)!-1$ ✓

d. $1-\dfrac{1}{2n+1}>0.999$ needs $2n+1>1000$, so $n=500$.

e. The first sum **converges to 1**; the second grows without limit (like a factorial), so it **diverges**.

## Answer to Question 18

a. $\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}\right)\left(x-\frac{x^3}{6}\right)=x+x^2+\frac{x^3}{2}-\frac{x^3}{6}+\cdots=x+x^2+\frac{1}{3}x^3+\cdots$

b. The numerator is $\frac{1}{3}x^3+\cdots$, so the limit is $\dfrac{1}{3}$.

c. $\ln(1+u)=u-\frac{u^2}{2}+\cdots$. $u=-\frac{x^2}{2}+\frac{x^4}{24}$ and $u^2=\frac{x^4}{4}+\cdots$. So
$\ln\cos x=-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^4}{8}=-\frac{x^2}{2}-\frac{x^4}{12}+\cdots$ ✓

d. $-0.005-0.0000083=-0.0050083$; calculator: $\ln(\cos 0.1)=-0.0050083$ ✓

e. $\dfrac{d}{dx}\ln\cos x=-\tan x$, so $-\tan x=-x-\frac{x^3}{3}$, giving $\tan x=x+\frac{1}{3}x^3+\cdots$.

## Answer to Question 19

a. Put $u=-x^2$ in $1+u+\frac{u^2}{2}+\frac{u^3}{6}$: $1-x^2+\frac{x^4}{2}-\frac{x^6}{6}+\cdots$ ✓

b. $a-\dfrac{a^3}{3}+\dfrac{a^5}{10}-\dfrac{a^7}{42}+\cdots$

c. $-0.041667$, $+0.003125$, $-0.000186$. Total **0.46127**.

d. Error $=0.461281-0.461272\approx 0.000009$. The next term is $+\dfrac{a^9}{216}=0.0000090$, which **matches the error**, as expected for an alternating series with decreasing terms.

e. Integrals like this have no formula in terms of standard functions, but the series converges quickly, so **values can be calculated to any accuracy** and printed in tables (or programmed into calculators).

## Answer to Question 20

a. $\dfrac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \cdots$ for $|t|$. Since $\displaystyle\int_0^x \frac{dt}{1+t^2} = \arctan x$, integrating term by term gives $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$ ✓

b. $\arctan 1 = \frac{\pi}{4}$, so $\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$. Four terms: $4 \times 0.72381 = \mathbf{2.8952}$, a poor estimate.

c. $\arctan \frac{1}{\sqrt{3}} = \frac{\pi}{6}$, and $\left(\frac{1}{\sqrt{3}}\right)^{2k+1} = \frac{1}{\sqrt{3}} \cdot \frac{1}{3^k}$. So $\frac{\pi}{6} = \frac{1}{\sqrt{3}}\left(1 - \frac{1}{3\times 3} + \frac{1}{5\times 9} - \frac{1}{7\times 27} + \cdots\right)$; multiply by 6 and use $\frac{6}{\sqrt{3}} = 2\sqrt{3}$ ✓

d. $1 - 0.111111 + 0.022222 - 0.005291 = 0.905820$, and $2\sqrt{3} = 3.464102$, so the estimate is $3.464102 \times 0.905820 = \mathbf{3.13785}$, within 0.004 of $\pi = 3.14159$.

e. In (b) the error after four terms is about 0.25, because the terms only shrink like $\frac{1}{n}$. In (c) each term is divided by an extra factor of 3, so the terms shrink geometrically and **the error after four terms is only about 0.004** (less than the next term, $\frac{2\sqrt{3}}{9\times 81} \approx 0.005$). A few more terms give many correct decimal places.