Test answers - Proof by induction

Year: Further Maths · Duration: 60 minutes · /20

Answer to Question 1 (4 marks)

$n = 1$: $1 \times 3 = 3$ and $\frac{1}{6} \times 1 \times 2 \times 9 = 3$ ✓ [1]. Assume $\sum_{r=1}^{k} r(r+2) = \frac{1}{6}k(k+1)(2k+7)$ [1]. Then the sum to $k+1$ is $\frac{1}{6}k(k+1)(2k+7) + (k+1)(k+3) = \frac{1}{6}(k+1)(2k^2+13k+18) = \frac{1}{6}(k+1)(k+2)(2k+9)$ [1], which is the formula for $n = k+1$. Since the result is true for $n = 1$, and if it is true for $n = k$ then it is true for $n = k+1$, by mathematical induction it is true for all positive integers n. [1]

The key algebra, after taking out $\frac{1}{6}(k+1)$:
$k(2k+7) + 6(k+3) = 2k^2 + 13k + 18 = (k+2)(2k+9)$.

Answer to Question 2 (4 marks)

$f(1) = 10 = 5 \times 2$ ✓ [1]. Assume $6^k + 4 = 5m$ [1]. Then $6^{k+1} + 4 = 6 \cdot 6^k + 4 = 6(5m-4) + 4 = 30m - 20$ [1] $= 5(6m-4)$, divisible by 5. Since the result is true for $n = 1$, and if it is true for $n = k$ then it is true for $n = k+1$, by mathematical induction it is true for all positive integers n. [1]

An alternative inductive step uses $f(k+1) - f(k) = 6^{k+1} - 6^k = 5 \times 6^k$, a multiple of 5, so $f(k+1) = f(k) + 5 \times 6^k$ is also a multiple of 5. Check $n = 2$: $36 + 4 = 40$ ✓.

Answer to Question 3 (4 marks)

$n = 1$: $\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$ ✓ [1]. Assume true for $n = k$.

$\begin{pmatrix} 1+2k & -4k \\ k & 1-2k \end{pmatrix}\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3+6k-4k & -4-8k+4k \\ 3k+1-2k & -4k-1+2k \end{pmatrix}$ [1]

$= \begin{pmatrix} 3+2k & -4-4k \\ k+1 & -1-2k \end{pmatrix} = \begin{pmatrix} 1+2(k+1) & -4(k+1) \\ k+1 & 1-2(k+1) \end{pmatrix}$ [1]. Since the result is true for $n = 1$, and if it is true for $n = k$ then it is true for $n = k+1$, by mathematical induction it is true for all positive integers n. [1]

Check $n = 2$: the formula gives $\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$, and squaring the matrix directly gives $9 - 4 = 5$, $-12 + 4 = -8$, $3 - 1 = 2$ and $-4 + 1 = -3$ ✓. Multiplying on the left instead of on the right is also valid here.

Answer to Question 4 (4 marks)

a. $u_2 = 17$, $u_3 = 65$ [1]
b. $u_1 = 4 + 1 = 5$ ✓ [1]. Assume $u_k = 4^k + 1$. Then $u_{k+1} = 4(4^k + 1) - 3 = 4^{k+1} + 1$ [1]. Since the result is true for $n = 1$, and if it is true for $n = k$ then it is true for $n = k + 1$, by mathematical induction it is true for all positive integers n. [1]

In (a), $u_2 = 4 \times 5 - 3 = 17$ and $u_3 = 4 \times 17 - 3 = 65$, which match $4^2 + 1$ and $4^3 + 1$. The key step is $4 \times 4^k + 4 - 3 = 4^{k+1} + 1$, where the assumption is used explicitly.

Answer to Question 5 (4 marks)

a. $2^4 = 16 = 4^2$; $2^5 = 32 > 25$ ✓ [1]
b. $n = 5$: $32 > 25$ ✓. Assume $2^k > k^2$ for some $k \geq 5$ [1]. Then $2^{k+1} = 2 \times 2^k > 2k^2$ [1] $> (k + 1)^2$. So true for $n = k + 1$, and by induction **true for all** $n \geq 5$ [1].

The base case is $n = 5$, not $n = 1$: the statement fails for $n = 2, 3, 4$, since for example $2^3 = 8 < 9$. The chain $2^{k+1} > 2k^2 > (k + 1)^2$ uses the assumption first and the given fact second. Check the given fact at $k = 5$: $50 > 36$ ✓. After 6 weeks, A has 64 followers and B has 36.