Worksheet - Proof by induction

Year: Further Maths · 20

Show your working.

Question 1

Question 1 – The sum of cubes *(Section A · Skills practice)*

Prove by induction that, for all positive integers n, $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$.

a. Show that the result is true for $n = 1$.
b. Write down the assumption for $n = k$.
c. Show that the result is true for $n = k + 1$.
d. Write a conclusion.
e. Use the result to find $1^3 + 2^3 + \cdots + 20^3$.

Question 2

Question 2 – A product sum *(Section A · Skills practice)*

Prove by induction that $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$ for all $n \in \mathbb{Z}^+$.

a. Prove the basis case.
b. Carry out the inductive step, showing your algebra clearly.
c. State the conclusion.
d. Hence evaluate $\sum_{r=11}^{20} r(r+1)$.

Question 3

Question 3 – Divisibility by 4 and by 8 *(Section A · Skills practice)*

Use induction to prove each statement for all positive integers n.

a. $5^n - 1$ is divisible by 4.
b. $3^{2n} - 1$ is divisible by 8.
c. Write $3^{2n} - 1$ as a power of 9 minus 1, and explain how the proof in part (b) is the same method as the proof in part (a) with 5 replaced by 9.

Question 4

Question 4 – A divisibility proof with a polynomial *(Section A · Skills practice)*

Let $f(n) = n^3 + 2n$.

a. Show that $f(k+1) - f(k) = 3(k^2 + k + 1)$.
b. Hence prove by induction that $f(n)$ is divisible by 3 for all positive integers n.
c. Check your result for $n = 10$.

Question 5

Question 5 – Powers of a matrix *(Section A · Skills practice)*

Let $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$.

a. Find $\mathbf{A}^2$ and $\mathbf{A}^3$.
b. Suggest a formula for $\mathbf{A}^n$.
c. Prove your formula by induction.
d. Hence find $\mathbf{A}^{10}$.

Question 6

Question 6 – A recurrence relation *(Section A · Skills practice)*

A sequence is defined by $u_1 = 2$ and $u_{n+1} = 3u_n + 2$.

a. Find u_2, u_3 and u_4.
b. Prove by induction that $u_n = 3^n - 1$.
c. Find the first term of the sequence that exceeds one million.

Question 7

Question 7 – Stacking cans in a supermarket *(Section B · Real-life practice)*

A supermarket builds a pyramid of cans with triangular layers. The top layer has 1 can, the next has 3, then 6, and in general layer r (counting from the top) has $\frac{1}{2}r(r+1)$ cans.

Layers n	1	2	3	4
Total cans T_n	1	4	10	

a. Complete the table.
b. Prove by induction that $T_n = \frac{1}{6}n(n+1)(n+2)$.
c. Find the number of cans in a pyramid with 10 layers.
d. The manager has 300 cans. Find the greatest number of complete layers she can build.

Question 8

Question 8 – A monthly savings plan *(Section B · Real-life practice)*

Ali saves £50 at the end of every month in an account that pays 1% interest per month. His balance after n months is u_n, where $u_1 = 50$ and $u_{n+1} = 1.01u_n + 50$.

a. Find u_2 and u_3.
b. Prove by induction that $u_n = 5000(1.01^n - 1)$.
c. Find his balance after two years.
d. How much of this balance is interest?

Question 9

Question 9 – Handshakes at a party *(Section B · Real-life practice)*

At a party every guest shakes hands once with every other guest. Let h_n be the number of handshakes when there are n guests.

a. Explain why $h_{n+1} = h_n + n$.
b. Prove by induction that $h_n = \frac{1}{2}n(n-1)$ for all $n \geq 2$.
c. Find the number of handshakes when 20 guests are present.
d. How many guests were present if there were 435 handshakes?

Question 10

Question 10 – The Tower of Hanoi *(Section B · Real-life practice)*

In the Tower of Hanoi puzzle, n discs of different sizes must be moved from one peg to another, one at a time, never placing a larger disc on a smaller one. The least number of moves m_n satisfies $m_1 = 1$ and $m_{n+1} = 2m_n + 1$.

a. Explain the recurrence relation: how can $n+1$ discs be moved using the solution for n discs?
b. Prove by induction that $m_n = 2^n - 1$.
c. Find the least number of moves for 10 discs.
d. A machine makes one move per second. How many days does it take to move 20 discs?

Question 11

Question 11 – Cutting a pizza *(Section B · Real-life practice)*

A pizza is cut with n straight cuts, each crossing every other cut, with no three cuts meeting at a point. The greatest number of pieces is R_n.

Cuts n	1	2	3	4
Pieces R_n	2	4	7	

a. Explain why the $(k+1)$th cut adds $k+1$ new pieces.

b. Prove by induction that $R_n = \frac{1}{2}(n^2 + n + 2)$.
c. Find the greatest number of pieces with 8 cuts.
d. Find the least number of cuts needed for at least 50 pieces.

Question 12

Question 12 – Compound against simple interest *(Section B · Real-life practice)*

A bank offers 10% interest per year. With compound interest £1 grows to 1.1^n pounds after n years; with simple interest it grows to $1 + 0.1n$ pounds.

a. Show that both give the same amount after 1 year.
b. Prove by induction that $(1 + x)^n \geq 1 + nx$ for all positive integers n and all $x \geq -1$ (Bernoulli's inequality).
c. Deduce that compound interest is never worse than simple interest at the same rate.
d. Find the difference between the two amounts for £2000 after 10 years.

Question 13

Question 13 – Tiling a courtyard *(Section B · Real-life practice)*

A square courtyard of 2^n by 2^n slabs has one slab removed for a fountain. A builder covers the remaining $4^n - 1$ slabs with L-shaped tiles, each covering exactly 3 slabs.

a. Show that the number of slabs to be covered is a whole number of L-tiles when $n = 1$, 2 and 3.
b. Prove by induction that $4^n - 1$ is divisible by 3 for all $n \geq 1$.
c. Find the number of L-tiles needed when $n = 5$.
d. Explain why divisibility alone does not prove that the tiling is possible, and describe how the board can be split into four quarters to build a full proof.

Question 14

Question 14 – A fractal snowflake *(Section B · Real-life practice)*

A Koch snowflake starts as an equilateral triangle of side 27 cm (stage 1). At each stage the middle third of every side is replaced by two sides of an equilateral bump, so each side becomes 4 sides that are one third as long.

Stage n	1	2	3
Number of sides s_n	3	12	
Side length (cm)	27	9	

a. Complete the table.
b. Explain why $s_{n+1} = 4s_n$, and prove by induction that $s_n = 3 \times 4^{n-1}$.
c. Show that the perimeter at stage n is $81\left(\frac{4}{3}\right)^{n-1}$ cm.

d. Find the perimeter at stage 5, and explain what happens to the perimeter as n increases.

Question 15

Question 15 – Repeated derivatives *(Section C · Problem solving)*

Let $f(x) = xe^x$, and write $f^{(n)}(x)$ for the nth derivative.

n	1	2	3
$f^{(n)}(x)$			

a. Complete the table.
b. Suggest a formula for $f^{(n)}(x)$.
c. Prove your formula by induction.
d. Find $f^{(5)}(0)$.
e. Use a similar method to find and prove a formula for the nth derivative of $g(x) = xe^{2x}$.

Question 16

Question 16 – Fibonacci numbers and matrices *(Section C · Problem solving)*

The Fibonacci numbers are $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$. Let $\mathbf{Q} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

n	0	1	2	3	4	5	6	7	8
F_n	0	1	1	2	3	5	8		

a. Complete the table.
b. Prove by induction that $\mathbf{Q}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$ for $n \geq 1$.
c. By taking determinants, prove Cassini's identity $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$.
d. Check Cassini's identity for $n = 6$.
e. Use $\mathbf{Q}^{2n} = \mathbf{Q}^n\mathbf{Q}^n$ to show that $F_{2n} = F_n(F_{n+1} + F_{n-1})$.

Question 17

Question 17 – Factorials beat powers *(Section C · Problem solving)*

The table compares $n!$ with 2^n for small values of n.

n	1	2	3	4	5	6
$n!$	1	2	6	24		
2^n	2	4	8	16		

a. Complete the table, and state the smallest value of n for which $n! > 2^n$.
b. Prove by induction that $n! > 2^n$ for all integers $n \geq 4$.
c. Explain why the basis case must be $n = 4$ and not $n = 1$.
d. Prove by induction that $3^n > n^3$ for all integers $n \geq 4$. (You may use the fact that $\left(1 + \frac{1}{n}\right)^3 \leq \left(\frac{5}{4}\right)^3$ for $n \geq 4$.)
e. A student says that both inequalities show "exponentials always win". Comment on the two results.

Question 18

Question 18 – Two series in a spreadsheet *(Section C · Problem solving)*

A student uses a spreadsheet to add up two series and notices patterns in the totals.

n	1	2	3	4
$\sum_{r=1}^{n} \frac{1}{r(r+1)}$	$\frac{1}{2}$	$\frac{2}{3}$		
$\sum_{r=1}^{n} r\,2^r$	2	10		

a. Complete the table.
b. Prove by induction that $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$.
c. Prove by induction that $\sum_{r=1}^{n} r\,2^r = (n-1)2^{n+1} + 2$.
d. Find the smallest n for which the first sum exceeds 0.99.
e. Explain what happens to the first sum as n becomes very large.

Question 19

Question 19 – Divisibility with two powers *(Section C · Problem solving)*

Let $f(n) = 2^{n+2} + 3^{2n+1}$.

n	1	2	3
$f(n)$			

a. Complete the table and show that each value is divisible by 7.
b. Show that $f(k+1) = 2f(k) + 7 \times 3^{2k+1}$.
c. Hence prove by induction that $f(n)$ is divisible by 7 for all $n \geq 1$.
d. Explain why it would also be valid to consider $f(k+1) - 9f(k)$, and find this expression.

Question 20

Question 20 – Challenge: proving de Moivre's theorem *(Section D · Challenge)*

De Moivre's theorem states that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

a. Prove the theorem by induction for all positive integers n.
b. Show that $(\cos\theta + i\sin\theta)^{-1} = \cos\theta - i\sin\theta$.
c. Hence prove the theorem for all negative integers n.
d. Use the theorem to find $(1+i)^8$ and check by expanding $(1+i)^2$.