Answers - Proof by induction

Year: Further Maths

Answer to Question 1

a. Left side $1^3 = 1$; right side $\frac{1}{4} \times 1 \times 4 = 1$. **True for** $n = 1$.

b. Assume $\sum_{r=1}^{k} r^3 = \frac{1}{4}k^2(k+1)^2$ for some positive integer k.

c. $\sum_{r=1}^{k+1} r^3 = \frac{1}{4}k^2(k+1)^2 + (k+1)^3 = \frac{1}{4}(k+1)^2\left(k^2 + 4(k+1)\right) = \frac{1}{4}(k+1)^2(k+2)^2$, which is the formula with $n = k+1$ ✓

d. Since the result is true for $n = 1$, and if it is true for $n = k$ then it is true for $n = k+1$, by mathematical induction it is true for all positive integers n.

e. $\frac{1}{4} \times 400 \times 441 =$ **44 100**

Answer to Question 2

a. $n = 1$: left $1 \times 2 = 2$, right $\frac{1}{3} \times 1 \times 2 \times 3 = 2$ ✓

b. Assume $\sum_{r=1}^{k} r(r+1) = \frac{1}{3}k(k+1)(k+2)$. Then $\sum_{r=1}^{k+1} r(r+1) = \frac{1}{3}k(k+1)(k+2) + (k+1)(k+2) = \frac{1}{3}(k+1)(k+2)(k+3)$, the formula with $n = k+1$ ✓

c. Since the result is true for $n = 1$, and if it is true for $n = k$ then it is true for $n = k+1$, by mathematical induction it is true for all positive integers n.

d. $\frac{1}{3}(20)(21)(22) - \frac{1}{3}(10)(11)(12) = 3080 - 440 =$ **2640**

Answer to Question 3

a. Let $f(n) = 5^n - 1$. $f(1) = 4$ ✓. Assume $f(k) = 4m$. Then $f(k+1) = 5^{k+1} - 1 = 5 \cdot 5^k - 1 = 5(f(k)+1) - 1 = 5f(k) + 4 = 4(5m+1)$, divisible by 4. Since the result is true for $n = 1$, and if it is true for $n = k$ then it is true for $n = k+1$, by mathematical induction it is true for all positive integers n.

b. Let $g(n) = 3^{2n} - 1 = 9^n - 1$. $g(1) = 8$ ✓. Assume $g(k) = 8m$. Then $g(k+1) = 9 \cdot 9^k - 1 = 9(g(k)+1) - 1 = 9g(k) + 8 = 8(9m+1)$, divisible by 8. **True for all** $n \geq 1$ by induction.

c. $3^{2n} = (3^2)^n = 9^n$, so $3^{2n} - 1 = 9^n - 1$. In (a) the key step was $f(k+1) = 5f(k) + 4$, where 4 is $5 - 1$; in (b) it was $g(k+1) = 9g(k) + 8$, where 8 is $9 - 1$. In general $a^n - 1$ is divisible by $a - 1$, by exactly the same argument.

Answer to Question 4

a. $f(k+1) = k^3 + 3k^2 + 3k + 1 + 2k + 2 = k^3 + 3k^2 + 5k + 3$, so $f(k+1) - f(k) = 3k^2 + 3k + 3 = 3(k^2 + k + 1)$ ✓

b. $f(1) = 3$ ✓. Assume $f(k) = 3m$. Then $f(k+1) = f(k) + 3(k^2 + k + 1) = 3(m + k^2 + k + 1)$, a multiple of 3. Since the result is true for $n = 1$, and if it is true for $n = k$ then it is true for $n = k + 1$, by mathematical induction it is true for all positive integers n.

c. $f(10) = 1000 + 20 = 1020 = 3 \times 340$ ✓

Answer to Question 5

a. $\mathbf{A}^2 = \begin{pmatrix} 4 & 0 \\ 3 & 1 \end{pmatrix}$, $\mathbf{A}^3 = \begin{pmatrix} 8 & 0 \\ 7 & 1 \end{pmatrix}$

b. $\mathbf{A}^n = \begin{pmatrix} 2^n & 0 \\ 2^n - 1 & 1 \end{pmatrix}$

c. $n = 1$: $\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$ ✓. Assume $\mathbf{A}^k = \begin{pmatrix} 2^k & 0 \\ 2^k - 1 & 1 \end{pmatrix}$. Then $\mathbf{A}^{k+1} = \mathbf{A}^k\mathbf{A} = \begin{pmatrix} 2^k \cdot 2 & 0 \\ 2(2^k - 1) + 1 & 1 \end{pmatrix} = \begin{pmatrix} 2^{k+1} & 0 \\ 2^{k+1} - 1 & 1 \end{pmatrix}$ ✓. Since the result is true for $n = 1$, and if it is true for $n = k$ then it is true for $n = k + 1$, by mathematical induction it is true for all positive integers n.

d. $\begin{pmatrix} 1024 & 0 \\ 1023 & 1 \end{pmatrix}$

Answer to Question 6

a. $u_2 = 8$, $u_3 = 26$, $u_4 = 80$

b. $u_1 = 3 - 1 = 2$ ✓. Assume $u_k = 3^k - 1$. Then $u_{k+1} = 3(3^k - 1) + 2 = 3^{k+1} - 3 + 2 = 3^{k+1} - 1$ ✓. Since the result is true for $n = 1$, and if it is true for $n = k$ then it is true for $n = k + 1$, by mathematical induction it is true for all positive integers n.

c. $3^n - 1 > 10^6$: $3^{12} = 531\,441$ and $3^{13} = 1\,594\,323$, so the first is $u_{13} = 1\,594\,322$.

Answer to Question 7

a. $T_4 = 10 + 10 = \mathbf{20}$

b. $n = 1$: $\frac{1}{6} \times 1 \times 2 \times 3 = 1$ ✓. Assume $T_k = \frac{1}{6}k(k+1)(k+2)$. Then $T_{k+1} = T_k + \frac{1}{2}(k+1)(k+2) = \frac{1}{6}(k+1)(k+2)(k+3)$ ✓. Since the result is true for $n = 1$, and if it is true for $n = k$ then it is true for $n = k + 1$, by mathematical induction it is true for all positive integers n.

c. $\frac{1}{6} \times 10 \times 11 \times 12 =$ **220 cans**
d. $T_{11} = 286$ and $T_{12} = 364$, so **11 layers** (using 286 cans).

Answer to Question 8

a. $u_2 = 50.50 + 50 =$ **£100.50**; $u_3 = 101.505 + 50 =$ **£151.51**
b. $u_1 = 5000 \times 0.01 = 50$ ✓. Assume $u_k = 5000(1.01^k - 1)$. Then $u_{k+1} = 1.01 \times 5000(1.01^k - 1) + 50 = 5000 \times 1.01^{k+1} - 5050 + 50 = 5000(1.01^{k+1} - 1)$ ✓. Since the result is true for $n = 1$, and if it is true for $n = k$ then it is true for $n = k + 1$, by mathematical induction it is true for all positive integers n.
c. $u_{24} = 5000(1.01^{24} - 1) = 5000 \times 0.269735 =$ **£1348.67**
d. He paid in $24 \times 50 = £1200$, so the interest is **£148.67**.

Answer to Question 9

a. When a new guest arrives, the n guests already there each shake hands with the new guest: n **new handshakes**.
b. $n = 2$: $h_2 = 1 = \frac{1}{2} \times 2 \times 1$ ✓. Assume $h_k = \frac{1}{2}k(k-1)$. Then $h_{k+1} = \frac{1}{2}k(k-1) + k = \frac{1}{2}k(k+1) = \frac{1}{2}(k+1)k$ ✓. Since the result is true for $n = 2$ and the step holds, it is **true for all** $n \geq 2$ by induction.
c. $\frac{1}{2} \times 20 \times 19 =$ **190**
d. $n(n-1) = 870$, $n^2 - n - 870 = 0$, $(n-30)(n+29) = 0$: **30 guests**.

Answer to Question 10

a. Move the top n discs to the spare peg (m_n moves), move the largest disc (1 move), then move the n discs on top of it (m_n moves): $2m_n + 1$ moves.
b. $m_1 = 2 - 1 = 1$ ✓. Assume $m_k = 2^k - 1$. Then $m_{k+1} = 2(2^k - 1) + 1 = 2^{k+1} - 1$ ✓. Since the result is true for $n = 1$, and if it is true for $n = k$ then it is true for $n = k + 1$, by mathematical induction it is true for all positive integers n.
c. $2^{10} - 1 =$ **1023**
d. $2^{20} - 1 = 1\,048\,575$ s; dividing by $86\,400$ s per day gives **about 12.1 days**.

Answer to Question 11

a. The new cut crosses the k earlier cuts at k different points, which divide it into $k + 1$ segments. **Each segment splits one piece into two**, adding $k + 1$ pieces. (So $R_4 = 7 + 4 = 11$.)
b. $R_1 = \frac{1}{2}(1 + 1 + 2) = 2$ ✓. Assume $R_k = \frac{1}{2}(k^2 + k + 2)$. Then $R_{k+1} = \frac{1}{2}(k^2 + k + 2) + k + 1 = \frac{1}{2}(k^2 + 3k + 4) = \frac{1}{2}\big((k+1)^2 + (k+1) + 2\big)$ ✓. Since the result is true for $n = 1$, and if it is true for $n = k$ then it is true for

$n = k + 1$, by mathematical induction it is true for all positive integers n.

c. $\frac{1}{2}(64 + 8 + 2) = \mathbf{37}$

d. $R_9 = 46$ and $R_{10} = 56$, so **10 cuts**.

Answer to Question 12

a. Both give **£1.10** for each pound.

b. $n = 1$: $1 + x \geq 1 + x$ ✓. Assume $(1 + kx) \leq (1 + x)^k$. Since $1 + x \geq 0$, multiply both sides by $1 + x$:
$(1 + x)^{k+1} \geq (1 + x)(1 + kx) = 1 + (k + 1)x + kx^2 \geq 1 + (k + 1)x$, because $kx^2 \geq 0$ ✓. Since the result is true for $n = 1$, and if it is true for $n = k$ then it is true for $n = k + 1$, by mathematical induction it is true for all positive integers n.

c. Take $x = 0.1$: $1.1^n \geq 1 + 0.1n$ for every n, so **compound interest gives at least as much**.

d. $2000(1.1^{10}) - 2000(2) = 5187.48 - 4000 = \mathbf{£1187.48}$

Answer to Question 13

a. $4 - 1 = 3$, $16 - 1 = 15$, $64 - 1 = 63$: **1, 5 and 21 tiles**.

b. $f(1) = 3$ ✓. Assume $4^k - 1 = 3m$. Then
$4^{k+1} - 1 = 4 \cdot 4^k - 1 = 4(3m + 1) - 1 = 12m + 3 = 3(4m + 1)$ ✓. Since the result is true for $n = 1$, and if it is true for $n = k$ then it is true for $n = k + 1$, by mathematical induction it is true for all positive integers n.

c. $\dfrac{4^5 - 1}{3} = \dfrac{1023}{3} = \mathbf{341}$

d. A count only shows that the area is right; the shape might still not fit. Split the 2^{k+1} board into four 2^k quarters. One quarter contains the fountain; place one L-tile at the centre covering one corner slab of each of the other three quarters. Now each quarter has exactly one slab missing, and **by the inductive hypothesis each can be tiled**.

Answer to Question 14

a. $s_3 = 48$, side length 3 cm.

b. Every side becomes 4 sides, so $s_{n+1} = 4s_n$. $s_1 = 3 \times 4^0 = 3$ ✓. Assume $s_k = 3 \times 4^{k-1}$; then $s_{k+1} = 4 \times 3 \times 4^{k-1} = 3 \times 4^k$ ✓. Since the result is true for $n = 1$, and if it is true for $n = k$ then it is true for $n = k + 1$, by mathematical induction it is true for all positive integers n.

c. The side length at stage n is $27 \times \left(\frac{1}{3}\right)^{n-1}$ (by a similar induction), so the perimeter is $3 \times 4^{n-1} \times 27 \times 3^{-(n-1)} = 81\left(\frac{4}{3}\right)^{n-1}$ ✓

d. $81 \times \left(\frac{4}{3}\right)^4 = 81 \times \frac{256}{81} = \mathbf{256\ cm}$. Since $\frac{4}{3} > 1$, **the perimeter increases**

without limit, even though the snowflake stays inside a fixed circle.

Answer to Question 15

a. $f'(x) = (x+1)e^x$, $f''(x) = (x+2)e^x$, $f'''(x) = (x+3)e^x$
b. $f^{(n)}(x) = (x+n)e^x$
c. $n = 1$: $f'(x) = e^x + xe^x = (x+1)e^x$ ✓. Assume $f^{(k)}(x) = (x+k)e^x$. Differentiate: $f^{(k+1)}(x) = e^x + (x+k)e^x = (x+k+1)e^x$ ✓. Since the result is true for $n = 1$, and if it is true for $n = k$ then it is true for $n = k+1$, by mathematical induction it is true for all positive integers n.
d. $f^{(5)}(0) = (0+5)e^0 =$ **5**
e. $g'(x) = (1+2x)e^{2x}$, $g''(x) = (4+4x)e^{2x}$: conjecture $g^{(n)}(x) = 2^{n-1}(2x+n)e^{2x}$. $n = 1$: $(2x+1)e^{2x}$ ✓. Assume true for k; then $g^{(k+1)} = 2^{k-1}\left(2 + 2(2x+k)\right)e^{2x} = 2^k(2x+k+1)e^{2x}$ ✓, so true for all n.

Answer to Question 16

a. $F_7 = 13$, $F_8 = 21$
b. $n = 1$: $\mathbf{Q} = \begin{pmatrix} F_2 & F_1 \\ F_1 & F_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ ✓. Assume true for k. Then $\mathbf{Q}^{k+1} = \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} F_{k+1}+F_k & F_{k+1} \\ F_k + F_{k-1} & F_k \end{pmatrix} = \begin{pmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{pmatrix}$ ✓. Since the result is true for $n = 1$, and if it is true for $n = k$ then it is true for $n = k+1$, by mathematical induction it is true for all positive integers n.
c. $\det \mathbf{Q} = -1$, so $\det(\mathbf{Q}^n) = (-1)^n$. But $\det(\mathbf{Q}^n) = F_{n+1}F_{n-1} - F_n^2$, hence $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ ✓
d. $F_7F_5 - F_6^2 = 13 \times 5 - 64 = 65 - 64 = 1 = (-1)^6$ ✓
e. The top-right entry of $\mathbf{Q}^{2n}$ is F_{2n}. In $\mathbf{Q}^n\mathbf{Q}^n$ it is $F_{n+1}F_n + F_nF_{n-1} = F_n(F_{n+1} + F_{n-1})$ ✓

Answer to Question 17

a. $5! = 120$, $6! = 720$; $2^5 = 32$, $2^6 = 64$. **Smallest n is 4** ($24 > 16$).
b. $n = 4$: $24 > 16$ ✓. Assume $k! > 2^k$ for some $k \geq 4$. Then $(k+1)! = (k+1)k! > (k+1)2^k > 2 \times 2^k = 2^{k+1}$, since $k+1 > 2$ ✓. Since true for $n = 4$ and true for $k+1$ whenever true for $k \geq 4$, **true for all $n \geq 4$** by induction.
c. For $n = 1, 2, 3$ the statement is false (1, 2, 6), so the induction must **start at the first value where it is true**.
d. $n = 4$: $81 > 64$ ✓. Assume $3^k > k^3$ for some $k \geq 4$. Then $3^{k+1} = 3 \times 3^k > 3k^3 > \left(1 + \frac{1}{k}\right)^3 k^3 = (k+1)^3$ ✓. **True for all $n \geq 4$.**
e. The first result shows that factorials eventually beat the exponential 2^n, so

exponentials do **not** always win; the second shows that the exponential 3^n beats the power n^3. Each comparison needs its own proof and its own starting value.

Answer to Question 18

a. First row: $\frac{3}{4}$, $\frac{4}{5}$. Second row: $10 + 24 = 34$, $34 + 64 = 98$.

b. $n = 1$: $\frac{1}{2} = \frac{1}{2}$ ✓. Assume the sum to k is $\frac{k}{k+1}$. Then the sum to $k+1$ is $\frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$ ✓. Since the result is true for $n = 1$, and if it is true for $n = k$ then it is true for $n = k+1$, by mathematical induction it is true for all positive integers n.

c. $n = 1$: $(0)2^2 + 2 = 2$ ✓. Assume the sum to k is $(k-1)2^{k+1} + 2$. Adding $(k+1)2^{k+1}$ gives $2k \cdot 2^{k+1} + 2 = k\,2^{k+2} + 2$, which is the formula with $n = k+1$ ✓. Since the result is true for $n = 1$, and if it is true for $n = k$ then it is true for $n = k+1$, by mathematical induction it is true for all positive integers n.

d. $\frac{n}{n+1} > 0.99$ gives $n > 0.99n + 0.99$, $0.01n > 0.99$, $n > 99$: $n = 100$.

e. $\frac{n}{n+1} = 1 - \frac{1}{n+1}$, which **tends to 1**: the infinite series converges to 1.

Answer to Question 19

a. $f(1) = 8 + 27 = 35 = 7 \times 5$, $f(2) = 16 + 243 = 259 = 7 \times 37$, $f(3) = 32 + 2187 = 2219 = 7 \times 317$.

b. $f(k+1) = 2^{k+3} + 3^{2k+3} = 2 \cdot 2^{k+2} + 9 \cdot 3^{2k+1} = 2\left(2^{k+2} + 3^{2k+1}\right) + 7 \cdot 3^{2k+1} = 2f(k) + 7 \times 3^{2k+1}$ ✓

c. $f(1) = 35$ ✓. Assume $f(k) = 7m$. Then $f(k+1) = 14m + 7 \times 3^{2k+1} = 7(2m + 3^{2k+1})$, divisible by 7 ✓. Since the result is true for $n = 1$, and if it is true for $n = k$ then it is true for $n = k+1$, by mathematical induction it is true for all positive integers n.

d. $f(k+1) - 9f(k) = 2 \cdot 2^{k+2} - 9 \cdot 2^{k+2} = -7 \times 2^{k+2}$, also a multiple of 7, so $f(k+1) = 9f(k) - 7 \times 2^{k+2}$ gives an equally good proof.

Answer to Question 20

a. $n = 1$ is clear. Assume $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$. Then $(\cos\theta + i\sin\theta)^{k+1} = (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta) = (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta) = \cos(k+1)\theta + i\sin(k+1)\theta$ by the addition formulae ✓. Since the result is true for $n = 1$, and if it is true for $n = k$ then it is true for $n = k+1$, by mathematical induction it is true for all positive integers n.

b. $(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = \cos^2\theta + \sin^2\theta = 1$, so the inverse is $\cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta)$ ✓

c. Let $n = -m$ with m a positive integer.
$(\cos\theta + i\sin\theta)^{-m} = \left((\cos\theta + i\sin\theta)^{-1}\right)^m = (\cos(-\theta) + i\sin(-\theta))^m = \cos(-m\theta) + i\sin(-m\theta)$
by part (a), which is $\cos n\theta + i\sin n\theta$ ✓

d. $1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$, so $(1+i)^8 = 16(\cos 2\pi + i\sin 2\pi) = \mathbf{16}$. Check:
$(1+i)^2 = 2i$, so $(1+i)^8 = (2i)^4 = 16i^4 = 16$ ✓