Test answers - Polar coordinates

Year: Further Maths · Duration: 60 minutes · /20

Answer to Question 1 (4 marks)

Method: use $x = r\cos\theta$, $y = r\sin\theta$ to go from polar to Cartesian, and $r = \sqrt{x^2 + y^2}$, $\tan\theta = \dfrac{y}{x}$ to go back. Always plot the point to choose the correct quadrant for θ. Mark scheme as shown in brackets.

a. $r = \sqrt{12 + 4} = 4$ [1]; second quadrant, $\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$: $(4, \frac{5\pi}{6})$ [1] Check: $4\cos\frac{5\pi}{6} = -2\sqrt{3}$ and $4\sin\frac{5\pi}{6} = 2$.

b. Multiply both sides by r so that r^2 and $r\sin\theta$ appear, then complete the square in y. $r^2 = 6r\sin\theta$: $x^2 + y^2 = 6y$ [1], i.e. $x^2 + (y-3)^2 = 9$: **circle, centre** $(0, 3)$**, radius 3** [1] The circle passes through the pole.

Answer to Question 2 (4 marks)

Method: the area swept out by a polar curve is $A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta$. Square r first, replace $\cos^2\theta$ or $\sin^2\theta$ by double angle expressions, then integrate term by term. Mark scheme as shown in brackets.

a. $\sin\theta$ takes values from -1 to 1. $r_{\max} = 5$ **at** $\theta = \frac{\pi}{2}$ [1]; $r_{\min} = 1$ **at** $\theta = \frac{3\pi}{2}$ [1] Since $r \geq 1 > 0$, the curve never passes through the pole.

b. Use $4\sin^2\theta = 2 - 2\cos 2\theta$; the sine and cosine terms integrate to 0 over a full turn. $\frac{1}{2}\int_0^{2\pi}(9 + 12\sin\theta + 4\sin^2\theta)\,d\theta = \frac{1}{2}\int_0^{2\pi}(11 + 12\sin\theta - 2\cos 2\theta)\,d\theta$ [1] $= \frac{1}{2} \times 22\pi = 11\pi$ [1] So the area is $11\pi \approx 34.6$ square units.

Answer to Question 3 (4 marks)

Method: the area swept out by a polar curve is $A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta$. Square r first, replace $\cos^2\theta$ or $\sin^2\theta$ by double angle expressions, then integrate term by term. Mark scheme as shown in brackets.

a. The tip is where r is largest. $\sin 2\theta = 1$ at $\theta = \frac{\pi}{4}$: $(4, \frac{\pi}{4})$ [1]

b. Use $\sin^2 2\theta = \frac{1}{2}(1 - \cos 4\theta)$. $A = \frac{1}{2}\int_0^{\pi/2} 16\sin^2 2\theta\,d\theta$ [1] $= 4\int_0^{\pi/2}(1 - \cos 4\theta)\,d\theta$ [1] $= 4\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/2} = 4 \times \frac{\pi}{2} = 2\pi$ ✓ [1] So the petal has area $2\pi \approx 6.28$.

Answer to Question 4 (4 marks)

Method: a tangent is parallel to the initial line where $\frac{dy}{d\theta} = 0$, with $y = r\sin\theta$. Write y in terms of θ, differentiate, and factorise using $\cos 2\theta = 2\cos^2\theta - 1$. Mark scheme as shown in brackets.

a. $y = 4(\sin\theta + \frac{1}{2}\sin 2\theta)$, $\frac{dy}{d\theta} = 4(\cos\theta + \cos 2\theta)$ [1] $= 4(2\cos\theta - 1)(\cos\theta + 1) = 0$ [1], so $\theta = \pm\frac{\pi}{3}$ [1]

b. $r = 6$: $(6\cos\frac{\pi}{3},\ 6\sin\frac{\pi}{3}) = (3,\ 3\sqrt{3})$ [1]

At $\theta = \pi$, $r = 0$: this is the cusp at the pole, which is why it is excluded.

Check: at $\theta = \frac{\pi}{3}$, $\cos\frac{\pi}{3} + \cos\frac{2\pi}{3} = \frac{1}{2} - \frac{1}{2} = 0$ ✓.

Answer to Question 5 (4 marks)

Method: the area swept out by a polar curve is $A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta$. Square r first, replace $\cos^2\theta$ or $\sin^2\theta$ by double angle expressions, then integrate term by term. For the region between two curves, use $\frac{1}{2}\int(r_1^2 - r_2^2)\,d\theta$ with r_1 the outer curve. Mark scheme as shown in brackets.

a. $4 = 4(1 - \cos\theta)$ gives $\cos\theta = 0$: $\theta = \pm\frac{\pi}{2}$ [1]

b. $A = 2 \times \frac{1}{2}\int_0^{\pi/2}\left(16 - 16(1-\cos\theta)^2\right)d\theta = 16\int_0^{\pi/2}(2\cos\theta - \cos^2\theta)\,d\theta$ [1];

$= 16\left[2\sin\theta - \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{\pi/2}$ [1] $= 16\left(2 - \frac{\pi}{4}\right) = 32 - 4\pi \approx 19.4$ **m²** ✓ [1]

By symmetry about the initial line, we integrate from 0 to $\frac{\pi}{2}$ and double.