Lesson - Polar coordinates

Year: Further Maths

Polar coordinates locate a point by its distance from a fixed point and an angle. They make spirals, petals and loops easy to describe, and give a neat formula for area.

Starter activity

Stand in the middle of the school field and describe the position of a football to a friend in two different ways. You could say "12 metres east and 5 metres north", or you could say "13 metres away, in the direction about 23 degrees north of east". Both descriptions locate the ball exactly. Which one would a lighthouse keeper, a radar operator or a sprinkler designer prefer, and why? Try writing the circle of radius 13 metres around you in both systems and decide which equation is simpler.

Polar and Cartesian coordinates

Definition: polar coordinates

A point is given by (r, θ): the distance r from the pole O and the angle θ anticlockwise from the initial line (the positive x-axis). A point with negative r is plotted at $(|r|, \theta + \pi)$.

Rule: conversions

$$x = r\cos\theta, \quad y = r\sin\theta, \qquad r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}.$$

Check the quadrant when finding θ.

Method: converting an equation

1. Replace x by $r\cos\theta$ and y by $r\sin\theta$ (or the reverse).
2. Multiply through by r if it helps create r^2 or $r\cos\theta$.
3. Simplify and identify the curve.

Example 1

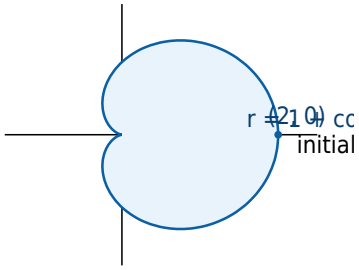
Convert $r = 4\cos\theta$ to Cartesian form.

$r^2 = 4r\cos\theta \Rightarrow x^2 + y^2 = 4x \Rightarrow (x-2)^2 + y^2 = 4$, a circle of radius 2 centred at $(2, 0)$. Similarly the line $x = 3$ is $r = 3\sec\theta$, and $y = \sqrt{3}x$ is $\theta = \frac{\pi}{3}$.

Sketching polar curves

Key fact: standard curves

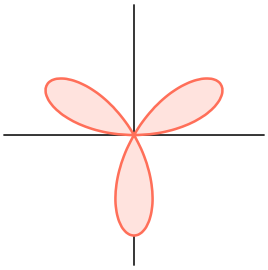
- $r = a$: circle, centre O. $r = 2a\cos\theta$: circle through O, centre $(a, 0)$. $r = 2a\sin\theta$: circle, centre $(0, a)$.
- $\theta = \alpha$: a line through O. $r = a\theta$: a spiral.
- Cardioid $r = a(1 + \cos\theta)$; limaçon $r = a + b\cos\theta$.
- Rose $r = a\cos n\theta$ or $a\sin n\theta$: n petals if n is odd, $2n$ if even. Lemniscate $r^2 = a^2\cos 2\theta$.



The cardioid r = 1 + cos θ. Its area is 3π/2.

Method: sketching a polar curve

1. Find the greatest and least values of r.
2. Find where $r = 0$; the curve is tangent to that line at the pole.
3. Use symmetry: $\cos\theta$ gives symmetry about the initial line, $\sin\theta$ about $\theta = \frac{\pi}{2}$.
4. Tabulate several values of θ and join the points.
5. Mark axis crossings and the points of greatest r.



The three-petalled rose r = sin 3θ.

Area

Rule: area enclosed by a polar curve

The area enclosed by $r = f(\theta)$ and the radii $\theta = \alpha$ and $\theta = \beta$ is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta.$$

For the area between two curves use $\frac{1}{2}\int(r_{\text{outer}}^2 - r_{\text{inner}}^2)\,d\theta$. Use symmetry to halve the work, but state it.

Example 2: cardioid

Find the area of $r = a(1+\cos\theta)$.

$A = \frac{1}{2}a^2\int_0^{2\pi}(1+2\cos\theta+\cos^2\theta)d\theta = \frac{1}{2}a^2(2\pi+0+\pi) = \frac{3}{2}\pi a^2$, using $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$.

Example 3: one petal

Find the area of one petal of $r = \sin 3\theta$.

One petal is $0 \le \theta \le \frac{\pi}{3}$. $A = \frac{1}{2}\int_0^{\pi/3}\sin^2 3\theta\,d\theta = \frac{1}{2}\cdot\frac{\pi}{6} = \frac{\pi}{12}$.

Example 4: limaçon

Describe and find the area of $r = 2+\cos\theta$.

The curve never reaches the pole: r varies from 1 (at $\theta=\pi$) to 3 (at $\theta=0$), and it is symmetric about the initial line. $A = \frac{1}{2}\int_0^{2\pi}(2+\cos\theta)^2 d\theta = \frac{9\pi}{2}$.

Watch out

- Forgetting the factor $\frac{1}{2}$, or not squaring r.
- Using limits that go through the pole twice or beyond a single loop.
- Finding intersections only by equating r-values: also check the pole separately, as it may be reached at different angles.
- Working in degrees; integration needs radians.

Tangents

Rule: horizontal and vertical tangents

With $x = r\cos\theta$ and $y = r\sin\theta$, the tangent is **parallel to the initial line** where $\frac{d}{d\theta}(r\sin\theta) = 0$, and **perpendicular to it** where $\frac{d}{d\theta}(r\cos\theta) = 0$.

Example 5

Find the greatest height of the cardioid $r = a(1+\cos\theta)$ above the initial line.

$y = a(1+\cos\theta)\sin\theta$ and $\frac{dy}{d\theta} = a(\cos\theta + \cos 2\theta) = a(2\cos\theta - 1)(\cos\theta + 1) = 0$. So $\theta = \frac{\pi}{3}$, giving $y = \frac{3\sqrt{3}}{4}a$.

Note: a rotated curve

$r = 1 + \sin\theta$ is the cardioid $r = 1 + \cos\theta$ rotated through 90° about the pole. It has the same area $\frac{3\pi}{2}$, but the highest point is now at $\theta = \frac{\pi}{2}$, the point $(0, 2)$.

More methods and worked examples

Method: the area between two polar curves

1. Sketch both curves on the same diagram, so that you can see which curve is outside in the region you want.
2. Find the values of θ at the points of intersection by equating the r-values, and check the pole separately.
3. For each interval of θ, write the area as $\frac{1}{2}\int \left(r_{\text{outer}}^2 - r_{\text{inner}}^2\right) d\theta$. Do not subtract the r-values before squaring.
4. Use symmetry about the initial line whenever possible: integrate from 0 and double the result.
5. Use the double angle formulae to integrate $\cos^2\theta$ and $\sin^2\theta$, and simplify exact values carefully.

Worked example 6: a straight line in polar form

Find the polar equation of the line $y = 2$ and the area of the region bounded by the line and the half-lines $\theta = \frac{\pi}{4}$ and $\theta = \frac{\pi}{2}$.

Since $y = r\sin\theta$, the line is $r\sin\theta = 2$, so $r = 2\operatorname{cosec}\theta$. The area is $\frac{1}{2}\int_{\pi/4}^{\pi/2} 4\operatorname{cosec}^2\theta\, d\theta = 2\left[-\cot\theta\right]_{\pi/4}^{\pi/2} = 2(0+1) = 2$. As a check, the region is the triangle with vertices at the pole, $(2, 2)$ and $(0, 2)$, which has area $\frac{1}{2}\times 2\times 2 = 2$. This shows that the polar area formula works for any region bounded by two half-lines from the pole, even when the boundary is straight.

Worked example 7: inside a cardioid and outside a circle

Find the area of the region inside $r = 2(1+\cos\theta)$ and outside $r = 3$.

The curves meet when $2 + 2\cos\theta = 3$, so $\cos\theta = \frac{1}{2}$ and $\theta = \pm\frac{\pi}{3}$. Between these angles the cardioid is outside the circle. Using symmetry, the area is $2\times\frac{1}{2}\int_0^{\pi/3}\left(4(1+\cos\theta)^2 - 9\right)d\theta = \int_0^{\pi/3}(-3 + 8\cos\theta + 2\cos 2\theta)\, d\theta$. This equals $\left[-3\theta + 8\sin\theta + \sin 2\theta\right]_0^{\pi/3} = -\pi + 4\sqrt{3} + \frac{\sqrt{3}}{2} = \frac{9}{2}\sqrt{3} - \pi$, which is about 4.65 square units. Notice that we squared each radius before subtracting: the expression

$(2 + 2\cos\theta - 3)^2$ would give a wrong answer.

Worked example 8: a tangent perpendicular to the initial line

Find the points on $r = 2 + \cos\theta$ where the tangent is perpendicular to the initial line.

We need $\dfrac{d}{d\theta}(r\cos\theta) = 0$. Here $x = 2\cos\theta + \cos^2\theta$, so $\dfrac{dx}{d\theta} = -2\sin\theta - 2\sin\theta\cos\theta = -2\sin\theta(1 + \cos\theta)$. This is zero when $\sin\theta = 0$ or $\cos\theta = -1$, that is at $\theta = 0$ and $\theta = \pi$. The points are $(3, 0)$ and $(1, \pi)$, the two ends of the curve on the initial line. Since $2 \geq 2 \times 1$, this limaçon is convex and has no dimple, so there are no other vertical tangents.

Watch out: negative values of r

Some textbooks allow negative values of r, plotting the point in the opposite direction, while examination boards usually restrict curves to $r \geq 0$. Always read the question carefully to see which convention is used. For example, $r = \cos 2\theta$ has four petals if negative values are plotted, but only two if they are not. When you find an area, use limits that trace each part of the curve exactly once: integrating over a whole period of a rose curve may count some petals twice or include parts where r is negative.

The family of limaçons

The curves $r = a + b\cos\theta$ with $a, b > 0$ are called limaçons, from an old word for a snail. Their shape depends on the ratio of a to b. When $a \geq 2b$ the curve is convex, like an egg. When b the curve has a dimple near $\theta = \pi$. When $a = b$ the curve is a cardioid with a cusp at the pole. When a, r becomes zero at two angles and the curve, if negative values are allowed, has an inner loop. In every case the curve is symmetrical about the initial line, because cosine is an even function, and its area is $\pi\left(a^2 + \frac{1}{2}b^2\right)$, as you can check by integration.

Exam technique for polar questions

Examiners give marks for a correct integral with correct limits, even when the final answer is wrong, so always write the integral in full before simplifying. Show the double angle identity that you use. When a question asks for an exact answer, keep π and surds; when it asks for a sketch, mark the values of r where the curve crosses the initial line and the half-line $\theta = \frac{\pi}{2}$, and show the behaviour at the pole. A neat sketch also helps you to choose the limits of the integral, which is where most errors in this topic occur.

Worked example 9: sketching a rose curve

Sketch the curve $r = 3\cos 2\theta$ for $r \geq 0$ and find the total area that it encloses.

The curve exists only where $\cos 2\theta \geq 0$, that is for $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$ and $\frac{3\pi}{4} \leq \theta \leq \frac{5\pi}{4}$. In each of these intervals r rises from 0 to a maximum of 3 and falls back to 0, so the curve consists of two petals, one pointing along the initial line and one pointing in the opposite direction. A table of values at intervals of $\frac{\pi}{12}$ helps to draw each petal smoothly. The area of one petal is $\frac{1}{2}\int_{-\pi/4}^{\pi/4} 9\cos^2 2\theta \, d\theta = \frac{9}{4}\int_{-\pi/4}^{\pi/4}(1+\cos 4\theta)\, d\theta = \frac{9}{4}\times\frac{\pi}{2} = \frac{9\pi}{8}$. The total area is therefore $\frac{9\pi}{4}$. If negative values of r were allowed, two further petals would appear along the half-lines $\theta = \pm\frac{\pi}{2}$, and the total area would double.

Why the area formula works

The formula for the area comes from thin sectors rather than thin rectangles. Divide the region into narrow slices, each with a small angle $\delta\theta$ at the pole. Each slice is almost a sector of a circle with radius r, and the area of such a sector is $\frac{1}{2}r^2\delta\theta$. Adding the slices and letting $\delta\theta$ tend to zero turns the sum into the integral $\frac{1}{2}\int r^2\, d\theta$. This explains why the angle must be in radians, and why the region must be bounded by half-lines from the pole: the formula measures area swept out by a rotating radius, in the same way as a lighthouse beam sweeps across the sea.

Did you know?

Polar coordinates were used by Isaac Newton in the 1670s, but the modern name and notation came later, from the Italian mathematician Gregorio Fontana in the eighteenth century. The spiral $r = a\theta$ is named after Archimedes, who studied it more than 2000 years ago, and the lemniscate was described by Jacob Bernoulli in 1694. Today polar coordinates are used every day in radar, sonar, satellite navigation and in the design of microphones, whose pick-up patterns are printed as cardioids on the box.

Otto's tip

For area problems use $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$ and $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$ straight away. Always draw the curve before choosing limits.