Worksheet - Polar coordinates

Year: Further Maths · 20

Show your working.

Question 1

Question 1 – Converting coordinates *(Section A · Skills practice)*

Give answers exactly where possible. Polar angles are measured in radians in the interval $-\pi$.

a. Convert the polar coordinates $(4, \frac{\pi}{3})$ to Cartesian form.
b. Convert $(2\sqrt{2}, \frac{3\pi}{4})$ to Cartesian form.
c. Convert the Cartesian point $(-3, 3\sqrt{3})$ to polar form.
d. Convert $(5, -12)$ to polar form, giving θ to 3 decimal places.

Question 2

Question 2 – From polar to Cartesian equations *(Section A · Skills practice)*

Use $x = r\cos\theta$, $y = r\sin\theta$ and $r^2 = x^2 + y^2$ to find a Cartesian equation for each curve, and describe the curve.

a. $r = 4\cos\theta$
b. $r = 2\sec\theta$
c. $r^2 \sin 2\theta = 8$
d. $r = \dfrac{2}{1 - \cos\theta}$

Question 3

Question 3 – From Cartesian to polar equations *(Section A · Skills practice)*

Write each Cartesian equation in polar form, simplifying as far as possible.

a. $x^2 + y^2 = 9$
b. $y = x$ for $x > 0$
c. $x^2 + y^2 = 6y$
d. $(x^2 + y^2)^2 = 4xy$

Question 4

Question 4 – Sketching a cardioid *(Section A · Skills practice)*

The curve C has polar equation $r = 2(1 + \cos\theta)$ for $0 \leq \theta$.

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r					

a. Complete the table of values.
b. Explain why the curve is symmetrical about the initial line.
c. State the greatest value of r and the value of θ at which the curve passes through the pole.
d. Sketch the curve.

Question 5

Question 5 – Areas enclosed by polar curves *(Section A · Skills practice)*

Use the formula $A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta$ in each part.

a. Show that the formula gives the correct area for the circle $r = 3$.
b. Find the area enclosed by the cardioid $r = 2(1 + \cos\theta)$.
c. Find the area of the region bounded by the spiral $r = \theta$, $0 \le \theta \le \pi$, and the initial line.

Question 6

Question 6 – Tangents to a cardioid *(Section A · Skills practice)*

The curve C has equation $r = 1 + \cos\theta$, $-\pi$.

a. Show that $\frac{d}{d\theta}(r\sin\theta) = 2\cos^2\theta + \cos\theta - 1$.
b. Find the polar coordinates of the points where the tangent to C is parallel to the initial line (other than the pole).
c. Find the polar coordinates of the points where the tangent is perpendicular to the initial line.

Question 7

Question 7 – Aircraft on a radar screen *(Section B · Real-life practice)*

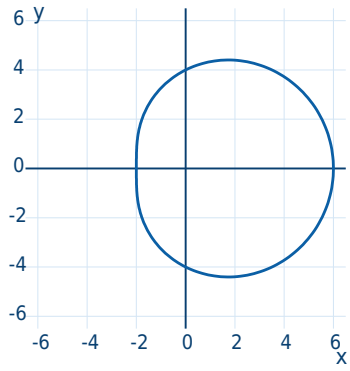
An air traffic controller uses polar coordinates centred on the radar station, with r in kilometres and θ measured anticlockwise from due east. Aircraft P is at $(40, \frac{\pi}{6})$ and aircraft Q is at $(30, \frac{2\pi}{3})$.

a. Find the Cartesian coordinates of P and Q, to 2 decimal places.
b. Find the angle $P\hat{O}Q$ between the two radar lines.
c. Find the distance between the two aircraft.
d. Aircraft must stay at least 5 km apart. Comment on the situation.

Question 8

Question 8 – A lawn sprinkler *(Section B · Real-life practice)*

A sprinkler stands at the pole O next to a wall. Because of the wind it waters the region bounded by the curve $r = 4 + 2\cos\theta$, where r is in metres.



a. State the greatest and least distances that the water reaches from the sprinkler.
b. Show that the area watered is 18π m².
c. The lawn is a rectangle of area 80 m², and the watered region lies entirely inside it. Find the percentage of the lawn that is watered.
d. Explain why the curve never passes through the pole.

Question 9

Question 9 – A coiled hose *(Section B · Real-life practice)*

A garden hose is coiled flat on the ground. Its centre line follows the spiral $r = \frac{1}{2}\theta$, $\theta \geq 0$, with r in decimetres.

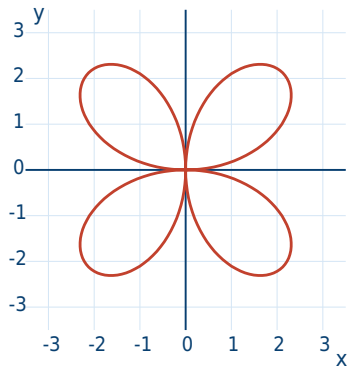
θ	$\frac{\pi}{2}$	π	2π	4π
r (dm)				

a. Complete the table, giving exact values.
b. Find the gap between successive turns of the spiral, measured along the initial line.
c. Find the area swept out by the radius as θ increases from 0 to 2π, to 3 significant figures.
d. The hose is 3 cm wide. Explain why neighbouring turns do not touch, and find the value of k for which the spiral $r = k\theta$ would make neighbouring turns of this hose just touch.

Question 10

Question 10 – A flower on a T-shirt *(Section B · Real-life practice)*

A design for a T-shirt is the four-petal rose $r = 3\sin 2\theta$, $0 \leq \theta$, where r is in centimetres. (Where r, the point is plotted in the opposite direction.)



a. Explain why one petal is traced as θ goes from 0 to $\frac{\pi}{2}$, and find the length of each petal.
b. Find the area of one petal.
c. Find the total area of the design.
d. Printing costs 2p per cm². Find the cost of printing the design, to the nearest penny.

Question 11

Question 11 – A microphone pick-up pattern *(Section B · Real-life practice)*

A cardioid microphone picks up sound best from the front. Its sensitivity in the direction θ is modelled by $r = 1 + \cos\theta$, where $\theta = 0$ points towards the singer.

a. Find the directions in which the sensitivity is 1.5, giving θ in degrees.
b. Find the area enclosed by the whole curve.
c. Show that the area of the part of the curve in front of the microphone ($-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$) is $\frac{3\pi}{4} + 2$.
d. Find the percentage of the total area that lies in front of the microphone, and explain why this type of microphone is useful on a noisy stage.

Question 12

Question 12 – A running track *(Section B · Real-life practice)*

A cross-country course around a lake follows the limaçon $r = 5 + 3\cos\theta$, with r in hundreds of metres and the pole at the start hut.

a. Find the distances from the hut to the points of the course on the initial line, and hence the length of the course measured along the initial line.
b. Find the area enclosed by the course, in m², to 3 significant figures.
c. A limaçon $r = a + b\cos\theta$ is convex when $a \ge 2b$. Is this course convex? Explain what this means for a runner.

Question 13

Question 13 – A lighthouse beam *(Section B · Real-life practice)*

A lighthouse O stands 4 km west of a straight north-south coastline. The beam turns anticlockwise from due east (the initial line).

a. Show that the coastline has polar equation $r = 4\sec\theta$ for $-\frac{\pi}{2}$.
b. Find the distance along the beam to the coast when $\theta = \frac{\pi}{3}$, and the Cartesian coordinates of the lit point.
c. Use $A = \frac{1}{2}\int r^2\,d\theta$ to find the area of sea swept by the beam between $\theta = 0$ and $\theta = \frac{\pi}{3}$.
d. Check your answer to part (c) using the area of a triangle.

Question 14

Question 14 – A satellite orbit *(Section B · Real-life practice)*

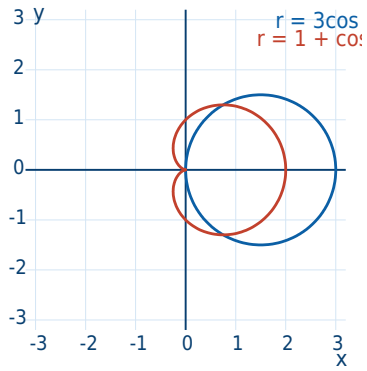
A satellite moves on an ellipse with the centre of the Earth at the pole. Its distance in kilometres is $r = \dfrac{8000}{1 + 0.2\cos\theta}$.

a. Find the least distance (perigee) and the greatest distance (apogee) of the satellite from the centre of the Earth.
b. Find r when $\theta = \frac{\pi}{2}$.
c. The radius of the Earth is 6370 km. Find the least height of the satellite above the surface.
d. Explain, using the equation, why the orbit is symmetrical about the initial line.

Question 15

Question 15 – A circle and a cardioid *(Section C · Problem solving)*

The diagram shows the circle C_1: $r = 3\cos\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, and the cardioid C_2: $r = 1 + \cos\theta$, $-\pi$.



a. Find the polar coordinates of the points of intersection of C_1 and C_2, apart from the pole.
b. Explain why the pole is on both curves, but is reached at different values of θ.
c. Shade, or describe, the region R that is inside C_1 and outside C_2.
d. Show that the area of R is exactly π.
e. Find the Cartesian equation of C_1 and confirm its area using the formula for a

circle.

Question 16

Question 16 – The dimple of a limaçon *(Section C · Problem solving)*

The curve L has polar equation $r = 3 + 2\cos\theta$, $-\pi$. The table compares some points of L.

θ	0	$\frac{\pi}{2}$	2.42	π
r	5	3		1
$x = r\cos\theta$	5	0		-1

a. Show that $\frac{d}{d\theta}(r\sin\theta) = 4\cos^2\theta + 3\cos\theta - 2$.

b. Find, to 3 decimal places, the value of θ in $(0, \pi)$ at which the tangent to L is parallel to the initial line, and find the distance of this tangent from the initial line.

c. Show that $\frac{d}{d\theta}(r\cos\theta) = -\sin\theta(3 + 4\cos\theta)$.

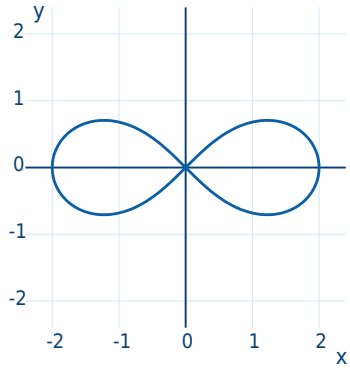
d. Find the points where the tangent is perpendicular to the initial line, and complete the table (take $\cos\theta = -\frac{3}{4}$ in the third column).

e. Use your table to explain why L has a dimple near $\theta = \pi$.

Question 17

Question 17 – The lemniscate of Bernoulli *(Section C · Problem solving)*

The curve with polar equation $r^2 = 4\cos 2\theta$ is called a lemniscate. It exists only where $\cos 2\theta \geq 0$.



a. Find the intervals of θ in $(-\pi, \pi]$ for which the curve exists, and state the greatest value of r.

b. Find the area of one loop and hence the total area enclosed.

c. Show that $y^2 = 4\cos 2\theta\sin^2\theta$ and that $\frac{d}{d\theta}\left(\cos 2\theta\sin^2\theta\right) = \sin 2\theta(2\cos 2\theta - 1)$.

d. Hence find the greatest height of the curve above the initial line.

e. Show that a Cartesian equation of the curve is $(x^2 + y^2)^2 = 4(x^2 - y^2)$.

Question 18

Question 18 – Between two turns of a spiral *(Section C · Problem solving)*

A snail shell pattern is modelled by the spiral $r = \theta$, $\theta \geq 0$, with r in millimetres.

Turn	θ from	θ to	r at end
First	0	2π	
Second	2π	4π	

a. Complete the table.
b. Find the area swept by the radius during the first turn.
c. Find the area swept during the second turn.
d. Explain why the area of the strip between the first and second turns is the difference of your answers to (c) and (b), and show that it equals $8\pi^3$ mm².
e. Show that the strip between the nth and $(n+1)$th turns has area $8n\pi^3$, so the strips grow in an arithmetic sequence.

Question 19

Question 19 – A three-petal logo *(Section C · Problem solving)*

A company logo is the curve $r = 2\cos 3\theta$, drawn only where $r \geq 0$, inside the circle $r = 2$.

θ	$-\frac{\pi}{6}$	$-\frac{\pi}{12}$	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$
r					

a. Complete the table and explain why it describes exactly one petal.
b. Find the angles of the tips of the three petals in $(-\pi, \pi]$.
c. Find the area of one petal and the total area of the logo.
d. Find the fraction of the circle $r = 2$ that is covered by the petals.
e. A designer wants a version with petals of area 3 cm² each. Find the value of a in $r = a\cos 3\theta$.

Question 20

Question 20 – Challenge: the cardioid in Cartesian form *(Section D · Challenge)*

The cardioid $r = a(1 + \cos\theta)$, $a > 0$, is the path traced by a point on a circle of radius $\frac{a}{2}$ rolling around another circle of the same size.

a. Show that the area enclosed is $\frac{3}{2}$ times the area of the circle $r = a$.
b. Show that a Cartesian equation of the cardioid is $(x^2 + y^2 - ax)^2 = a^2(x^2 + y^2)$.
c. Find the greatest distance of the curve from the initial line.
d. Find the area of the region inside the cardioid but outside the circle $r = a$.