Answers - Polar coordinates

Year: Further Maths

Answer to Question 1

Method: use $x = r\cos\theta$, $y = r\sin\theta$ to go from polar to Cartesian, and $r = \sqrt{x^2+y^2}$, $\tan\theta = \dfrac{y}{x}$ to go back. Always plot the point to choose the correct quadrant for θ.

a. $x = 4\cos\frac{\pi}{3} = 4\times\frac{1}{2} = 2$ and $y = 4\sin\frac{\pi}{3} = 4\times\frac{\sqrt{3}}{2} = 2\sqrt{3}$: $(2, 2\sqrt{3})$ This point is in the first quadrant, as expected for $\theta = \frac{\pi}{3}$.

b. $x = 2\sqrt{2}\cos\frac{3\pi}{4} = 2\sqrt{2}\times(-\frac{\sqrt{2}}{2}) = -2$ and $y = 2\sqrt{2}\times\frac{\sqrt{2}}{2} = 2$: $(-2, 2)$ This point is in the second quadrant.

c. $r = \sqrt{9+27} = \sqrt{36} = 6$. The point is in the second quadrant, so $\theta = \pi - \arctan\dfrac{3\sqrt{3}}{3} = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$: $(6, \frac{2\pi}{3})$ Check: $6\cos\frac{2\pi}{3} = -3$ and $6\sin\frac{2\pi}{3} = 3\sqrt{3}$.

d. $r = \sqrt{25+144} = 13$. The point is in the fourth quadrant, so $\theta = -\arctan\frac{12}{5} = -1.176$: $(13, -1.176)$ A calculator gives $\arctan\frac{-12}{5}$ directly in the fourth quadrant here.

Answer to Question 2

a. Multiply by r: $r^2 = 4r\cos\theta$, so $x^2+y^2 = 4x$, i.e. $(x-2)^2+y^2 = 4$: a **circle with centre $(2,0)$ and radius 2**.

b. $r\cos\theta = 2$, so $x = 2$: a **vertical straight line**.

c. $\sin 2\theta = 2\sin\theta\cos\theta$, so $2(r\sin\theta)(r\cos\theta) = 8$, i.e. $2xy = 8$: $xy = 4$, a **rectangular hyperbola**.

d. $r - r\cos\theta = 2$, so $r = x+2$. Square: $x^2+y^2 = x^2+4x+4$, so $y^2 = 4x+4$, a **parabola** with vertex $(-1, 0)$.

Answer to Question 3

Method: replace x^2+y^2 by r^2, x by $r\cos\theta$ and y by $r\sin\theta$, then divide by any common power of r (the pole is usually still on the curve).

a. $r^2 = 9$, so $r = 3$: a circle centred at the pole.

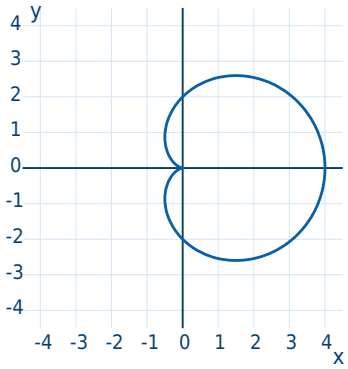
b. $\tan\theta = 1$ with $x > 0$: $\theta = \frac{\pi}{4}$, a half-line from the pole.

c. $r^2 = 6r\sin\theta$, so $r = 6\sin\theta$ (a circle through the pole with centre $(0, 3)$).

d. $r^4 = 4r^2\sin\theta\cos\theta = 2r^2\sin 2\theta$, so $r^2 = 2\sin 2\theta$.

Answer to Question 4

a. $r = 4,\ 3,\ 2,\ 1,\ 0$
b. $\cos(-\theta) = \cos\theta$, so the points (r, θ) and $(r, -\theta)$ are both on the curve: **the curve is its own reflection in the initial line**.
c. $\cos\theta \leq 1$, so $r_{\max} = 4$ **at** $\theta = 0$; $r = 0$ when $\cos\theta = -1$, i.e. $\theta = \pi$.
d. The curve is a **cardioid** (heart shape) with its cusp at the pole:



Answer to Question 5

Method: the area swept out by a polar curve is $A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta$. Square r first, replace $\cos^2\theta$ or $\sin^2\theta$ by double angle expressions, then integrate term by term.

a. Here $r^2 = 9$ is constant and a full turn is $0 \leq \theta \leq 2\pi$.
$A = \frac{1}{2}\int_0^{2\pi} 9\, d\theta = \frac{1}{2} \times 9 \times 2\pi = 9\pi = \pi \times 3^2$ ✓ The formula agrees with πr^2.
b. The cardioid is traced once as θ goes from 0 to 2π. Use $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$. $r^2 = 4(1 + 2\cos\theta + \cos^2\theta) = 4\left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right)$. Integrating from 0 to 2π the cosine terms give 0, so $A = \frac{1}{2} \times 4 \times \frac{3}{2} \times 2\pi = 6\pi$. So the cardioid encloses $6\pi \approx 18.8$ square units.
c. The spiral and the initial line enclose the region swept as θ goes from 0 to π.
$A = \frac{1}{2}\int_0^{\pi} \theta^2\, d\theta = \frac{1}{2}\left[\frac{\theta^3}{3}\right]_0^{\pi} = \frac{\pi^3}{6} \approx 5.17$ So the area is $\frac{\pi^3}{6}$.

Answer to Question 6

a. $r\sin\theta = \sin\theta + \sin\theta\cos\theta = \sin\theta + \frac{1}{2}\sin 2\theta$. Differentiate:
$\cos\theta + \cos 2\theta = \cos\theta + 2\cos^2\theta - 1$ ✓
b. $2\cos^2\theta + \cos\theta - 1 = (2\cos\theta - 1)(\cos\theta + 1) = 0$, so $\cos\theta = \frac{1}{2}$ (or $\cos\theta = -1$, which is the pole). $\theta = \pm\frac{\pi}{3}$ and $r = \frac{3}{2}$: $\left(\frac{3}{2}, \frac{\pi}{3}\right)$ **and** $\left(\frac{3}{2}, -\frac{\pi}{3}\right)$.
c. $r\cos\theta = \cos\theta + \cos^2\theta$, derivative $-\sin\theta - 2\sin\theta\cos\theta = -\sin\theta(1 + 2\cos\theta) = 0$.
$\sin\theta = 0$: $\theta = 0$, $r = 2$. $\cos\theta = -\frac{1}{2}$: $\theta = \pm\frac{2\pi}{3}$, $r = \frac{1}{2}$. $(2, 0)$**,** $\left(\frac{1}{2}, \frac{2\pi}{3}\right)$ **and** $\left(\frac{1}{2}, -\frac{2\pi}{3}\right)$

($\theta = \pi$ gives the pole, where the curve has a cusp).

Answer to Question 7

a. P: $(40\cos\frac{\pi}{6},\ 40\sin\frac{\pi}{6}) = (34.64,\ 20)$. Q: $(30\cos\frac{2\pi}{3},\ 30\sin\frac{2\pi}{3}) = (-15,\ 25.98)$.
b. $\frac{2\pi}{3} - \frac{\pi}{6} = \frac{\pi}{2}$, a right angle.
c. By the cosine rule, $PQ^2 = 40^2 + 30^2 - 2\times 40\times 30\cos\frac{\pi}{2} = 1600 + 900 - 0 = 2500$, so $PQ = 50$ **km** (or use Pythagoras since the angle is 90°).
d. The aircraft are 50 km apart, well above 5 km, so **they are safely separated** at the moment.

Answer to Question 8

a. $\cos\theta$ ranges from -1 to 1, so **greatest 6 m** (at $\theta = 0$) and **least 2 m** (at $\theta = \pi$).
b. $A = \frac{1}{2}\int_0^{2\pi}(16 + 16\cos\theta + 4\cos^2\theta)\,d\theta$. With $4\cos^2\theta = 2 + 2\cos 2\theta$:
$A = \frac{1}{2}\int_0^{2\pi}(18 + 16\cos\theta + 2\cos 2\theta)\,d\theta = \frac{1}{2}\times 18\times 2\pi = 18\pi$ ✓ (about 56.5 m²).
c. $\dfrac{18\pi}{80}\times 100 =$ **70.7%**
d. $r = 4 + 2\cos\theta \geq 4 - 2 = 2 > 0$ for every θ, so r **is never zero**.

Answer to Question 9

a. $r = \frac{\pi}{4},\ \frac{\pi}{2},\ \pi,\ 2\pi$
b. Along the initial line the spiral crosses at $\theta = 2\pi, 4\pi, \ldots$, where $r = \pi, 2\pi, \ldots$. The gap is π **dm** ≈ 3.14 dm.
c. $A = \frac{1}{2}\int_0^{2\pi}\frac{1}{4}\theta^2\,d\theta = \frac{1}{8}\left[\frac{\theta^3}{3}\right]_0^{2\pi} = \dfrac{8\pi^3}{24} = \dfrac{\pi^3}{3} \approx$ **10.3 dm²**.
d. The gap between turns is π dm ≈ 31.4 cm, which is much more than the 3 cm width, so **the turns do not touch**. For $r = k\theta$ the gap is $2\pi k$ dm. Setting $2\pi k = 0.3$ dm gives $k = \dfrac{0.3}{2\pi} \approx 0.0477$ dm per radian.

Answer to Question 10

a. $\sin 2\theta \geq 0$ for $0 \leq \theta \leq \frac{\pi}{2}$, starting and ending at $r = 0$: one petal. The greatest value $r = 3$ occurs at $\theta = \frac{\pi}{4}$, so each petal is **3 cm long**.
b. $A = \frac{1}{2}\int_0^{\pi/2} 9\sin^2 2\theta\,d\theta = \frac{9}{4}\int_0^{\pi/2}(1 - \cos 4\theta)\,d\theta = \frac{9}{4}\times\frac{\pi}{2} = \dfrac{9\pi}{8}$ **cm²** ≈ 3.53 cm²

c. $4 \times \dfrac{9\pi}{8} = \dfrac{9\pi}{2} \approx 14.1$ **cm²**

d. $14.137 \times 2 = 28.3$p, so about **28p**.

Answer to Question 11

Method: the area swept out by a polar curve is $A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta$. Square r first, replace $\cos^2\theta$ or $\sin^2\theta$ by double angle expressions, then integrate term by term.

a. Solve the equation $1 + \cos\theta = 1.5$ for $-180°$. $1 + \cos\theta = 1.5$ gives $\cos\theta = 0.5$: $\theta = \pm 60°$. So the sensitivity is 1.5 at $60°$ either side of the singer.

b. Expand r^2 and use $\int_0^{2\pi} \cos^2\theta\, d\theta = \pi$.
$A = \frac{1}{2}\displaystyle\int_0^{2\pi} (1 + 2\cos\theta + \cos^2\theta)\, d\theta = \tfrac{1}{2}(2\pi + 0 + \pi) = \dfrac{3\pi}{2}$ So the whole curve encloses $\frac{3\pi}{2} \approx 4.71$ square units.

c. $\frac{1}{2}\displaystyle\int_{-\pi/2}^{\pi/2} \left(\tfrac{3}{2} + 2\cos\theta + \tfrac{1}{2}\cos 2\theta\right) d\theta = \tfrac{1}{2}\left[\tfrac{3}{2}\theta + 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_{-\pi/2}^{\pi/2} = \tfrac{1}{2}\left(\tfrac{3\pi}{2} + 4 + 0\right) = \dfrac{3\pi}{4} + 2$
✓ This is the required result, about 4.36.

d. $\dfrac{3\pi/4 + 2}{3\pi/2} = \dfrac{1}{2} + \dfrac{4}{3\pi} = 0.924$, so **about 92%** lies in front: the microphone mostly ignores sound from the speakers behind it, which reduces feedback.

Answer to Question 12

a. $\theta = 0$: $r = 8$; $\theta = \pi$: $r = 2$, on the other side of the hut. Total $8 + 2 =$ **10, i.e. 1000 m**.

b. $A = \frac{1}{2}\displaystyle\int_0^{2\pi} (25 + 30\cos\theta + 9\cos^2\theta)\, d\theta = \tfrac{1}{2}(50\pi + 0 + 9\pi) = \dfrac{59\pi}{2} \approx 92.7$ (hundreds of m)². Each unit is $100 \times 100 = 10\,000$ m², so the area is about **927 000 m²**.

c. $a = 5$, $2b = 6$, and 5, so the course is **not convex**: it has a slight dent (dimple) near $\theta = \pi$, where the runner turns slightly the other way.

Answer to Question 13

a. The coast is the line $x = 4$, i.e. $r\cos\theta = 4$, so $r = \dfrac{4}{\cos\theta} = 4\sec\theta$ ✓

b. $r = 4\sec\frac{\pi}{3} = 4 \times 2 =$ **8 km**; the point is $(8\cos\frac{\pi}{3},\ 8\sin\frac{\pi}{3}) = (4,\ 4\sqrt{3})$.

c. $A = \frac{1}{2}\displaystyle\int_0^{\pi/3} 16\sec^2\theta\, d\theta = 8[\tan\theta]_0^{\pi/3} = 8\sqrt{3} \approx 13.9$ **km²**

d. The region is a right-angled triangle with vertices $(0,0)$, $(4,0)$ and $(4,4\sqrt{3})$: area

$\frac{1}{2} \times 4 \times 4\sqrt{3} = 8\sqrt{3}$ ✓

Answer to Question 14

a. r is least when $\cos\theta = 1$: $r = \dfrac{8000}{1.2} =$ **6667 km** (perigee). r is greatest when $\cos\theta = -1$: $r = \dfrac{8000}{0.8} =$ **10 000 km** (apogee).
b. $\cos\frac{\pi}{2} = 0$, so $r = 8000$ **km**.
c. $6667 - 6370 =$ **about 297 km**.
d. $\cos(-\theta) = \cos\theta$, so (r, θ) and $(r, -\theta)$ both lie on the orbit: **reflection in the initial line maps the orbit to itself**.

Answer to Question 15

a. $3\cos\theta = 1 + \cos\theta$ gives $\cos\theta = \frac{1}{2}$, so $\theta = \pm\frac{\pi}{3}$ and $r = \frac{3}{2}$: $(\frac{3}{2}, \frac{\pi}{3})$ **and** $(\frac{3}{2}, -\frac{\pi}{3})$.
b. C_1 passes through the pole at $\theta = \pm\frac{\pi}{2}$ and C_2 at $\theta = \pi$. Equating r-values only finds intersections at **the same** θ, so the pole must be checked separately.
c. R is the crescent between the two intersection points, for $-\frac{\pi}{3} \le \theta \le \frac{\pi}{3}$, where the circle is outside the cardioid.
d. By symmetry $A = 2 \times \frac{1}{2}\displaystyle\int_0^{\pi/3} \left(9\cos^2\theta - (1+\cos\theta)^2\right) d\theta = \int_0^{\pi/3} (8\cos^2\theta - 2\cos\theta - 1)\, d\theta$. With $8\cos^2\theta = 4 + 4\cos 2\theta$:
$\displaystyle\int_0^{\pi/3} (3 + 4\cos 2\theta - 2\cos\theta)\, d\theta = [3\theta + 2\sin 2\theta - 2\sin\theta]_0^{\pi/3} = \pi + \sqrt{3} - \sqrt{3} = \pi$ ✓
e. $r^2 = 3r\cos\theta$ gives $x^2 + y^2 = 3x$, i.e. $(x - \frac{3}{2})^2 + y^2 = \frac{9}{4}$: radius $\frac{3}{2}$, area $\frac{9}{4}\pi$.
Check: $\frac{1}{2}\displaystyle\int_{-\pi/2}^{\pi/2} 9\cos^2\theta\, d\theta = \frac{9}{2} \times \frac{\pi}{2} = \frac{9\pi}{4}$ ✓

Answer to Question 16

a. $r\sin\theta = 3\sin\theta + \sin 2\theta$, so the derivative is $3\cos\theta + 2\cos 2\theta = 3\cos\theta + 4\cos^2\theta - 2$ ✓
b. $\cos\theta = \dfrac{-3 + \sqrt{9 + 32}}{8} = \dfrac{-3 + \sqrt{41}}{8} = 0.4254$ (the other root is less than -1).
$\theta = 1.131$. Then $r = 3 + 2(0.4254) = 3.851$ and the tangent is $y = r\sin\theta = 3.851 \times 0.9050 =$ **3.485** units from the initial line.
c. $r\cos\theta = 3\cos\theta + 2\cos^2\theta$, derivative $-3\sin\theta - 4\sin\theta\cos\theta = -\sin\theta(3 + 4\cos\theta)$ ✓
d. $\sin\theta = 0$: $\theta = 0$ $(r = 5)$ and $\theta = \pi$ $(r = 1)$. $\cos\theta = -\frac{3}{4}$: $\theta = \pm 2.419$, $r = 3 - \frac{3}{2} = 1.5$ and $x = 1.5 \times (-\frac{3}{4}) = -1.125$. Points: $(5, 0)$, $(1, \pi)$, $(1.5, \pm 2.419)$.

e. At $\theta = \pi$ the curve is at $x = -1$, but at $\theta = \pm 2.419$ it reaches further left, $x = -1.125$. So the leftmost part of the curve bends inwards at $\theta = \pi$: **a dimple** (this agrees with $a = 3$).

Answer to Question 17

a. $\cos 2\theta \geq 0$ when $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$ or $\frac{3\pi}{4} \leq \theta \leq \pi$, $-\pi$. $r_{\max} = 2$ at $\theta = 0$ and $\theta = \pi$.

b. One loop: $\frac{1}{2}\int_{-\pi/4}^{\pi/4} 4\cos 2\theta \, d\theta = [\sin 2\theta]_{-\pi/4}^{\pi/4} = 1 - (-1) = \mathbf{2}$. Total area **4**.

c. $y = r\sin\theta$, so $y^2 = r^2\sin^2\theta = 4\cos 2\theta \sin^2\theta$ ✓. Product rule: $-2\sin 2\theta \sin^2\theta + \cos 2\theta \cdot 2\sin\theta\cos\theta = \sin 2\theta(\cos 2\theta - 2\sin^2\theta) = \sin 2\theta(\cos 2\theta - (1 - \cos 2\theta)) = \sin 2\theta(2\cos 2\theta - 1)$ ✓

d. The maximum of y in $(0, \frac{\pi}{4})$ needs $\cos 2\theta = \frac{1}{2}$, so $\theta = \frac{\pi}{6}$ and $r^2 = 2$, $r = \sqrt{2}$. Height $y = \sqrt{2}\sin\frac{\pi}{6} = \dfrac{\sqrt{2}}{2} \approx 0.707$.

e. $r^2 = 4(\cos^2\theta - \sin^2\theta)$. Multiply by r^2: $r^4 = 4(r^2\cos^2\theta - r^2\sin^2\theta)$, so $(x^2 + y^2)^2 = 4(x^2 - y^2)$ ✓

Answer to Question 18

a. $r = 2\pi$ and $r = 4\pi$

b. $\frac{1}{2}\displaystyle\int_0^{2\pi} \theta^2 \, d\theta = \frac{1}{6}(2\pi)^3 = \frac{4\pi^3}{3}$

c. $\frac{1}{2}\displaystyle\int_{2\pi}^{4\pi} \theta^2 \, d\theta = \frac{1}{6}(64\pi^3 - 8\pi^3) = \frac{28\pi^3}{3}$

d. The second sweep covers the region inside the second turn, which includes the region inside the first turn; removing that leaves the strip. $\dfrac{28\pi^3}{3} - \dfrac{4\pi^3}{3} = 8\pi^3 \approx 248$ **mm²** ✓

e. The sweep for turn $n+1$ is $\frac{1}{6}\left((2n+2)^3 - (2n)^3\right)\pi^3$ and for turn n is $\frac{1}{6}\left((2n)^3 - (2n-2)^3\right)\pi^3$. Their difference is $\frac{8\pi^3}{6}\left((n+1)^3 - 2n^3 + (n-1)^3\right) = \frac{8\pi^3}{6} \times 6n = 8n\pi^3$, which increases by $8\pi^3$ each time: an arithmetic sequence.

Answer to Question 19

a. $r = 0,\ \sqrt{2},\ 2,\ \sqrt{2},\ 0$. r starts and ends at 0 and is positive in between: **one petal**, centred on $\theta = 0$.

b. $r = 2$ when $\cos 3\theta = 1$: $3\theta = 0, \pm 2\pi$, so $\theta = 0$, $\pm\frac{2\pi}{3}$.

c. $\frac{1}{2}\int_{-\pi/6}^{\pi/6} 4\cos^2 3\theta\,d\theta = \int_{-\pi/6}^{\pi/6}(1+\cos 6\theta)\,d\theta = \frac{\pi}{3}+0 = \frac{\pi}{3}$. Total π.

d. $\dfrac{\pi}{4\pi} = \dfrac{1}{4}$

e. The area of a petal is proportional to a^2: $\dfrac{a^2}{4}\times\dfrac{\pi}{3} = 3$, so $a^2 = \dfrac{36}{\pi}$ and $a = \dfrac{6}{\sqrt{\pi}} \approx 3.39$ **cm**.

Answer to Question 20

a. $A = \frac{1}{2}a^2\int_0^{2\pi}\left(\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta\right)d\theta = \frac{1}{2}a^2\times 3\pi = \frac{3}{2}\pi a^2$. The circle $r=a$ has area πa^2, so the **ratio is** $\frac{3}{2}$ ✓

b. $r = a+a\cos\theta$; multiply by r: $r^2 = ar+ax$, so $r^2-ax = ar$. Square: $(x^2+y^2-ax)^2 = a^2r^2 = a^2(x^2+y^2)$ ✓

c. $y = a(1+\cos\theta)\sin\theta$, $\dfrac{dy}{d\theta} = a(\cos\theta+\cos 2\theta) = a(2\cos\theta-1)(\cos\theta+1) = 0$ gives $\theta = \frac{\pi}{3}$ and $y = a\cdot\frac{3}{2}\cdot\frac{\sqrt{3}}{2} = \dfrac{3\sqrt{3}}{4}a$.

d. The curves meet where $\cos\theta = 0$, $\theta = \pm\frac{\pi}{2}$. For $|\theta|$ the cardioid is outside the circle. Area $= 2\times\frac{1}{2}\int_0^{\pi/2} a^2\left((1+\cos\theta)^2-1\right)d\theta = a^2\int_0^{\pi/2}\left(2\cos\theta+\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)d\theta = a^2\left(2+\frac{\pi}{4}\right)$: $\left(2+\dfrac{\pi}{4}\right)a^2$.