Test answers - Matrices and linear transformations

Year: Further Maths · Duration: 60 minutes · /20

Answer to Question 1 (4 marks)

a. $\mathbf{AB} = \begin{pmatrix} 2+2 & 0-5 \\ 3-8 & 0+20 \end{pmatrix}$ [1] $= \begin{pmatrix} 4 & -5 \\ -5 & 20 \end{pmatrix}$ [1]

b. $\det \mathbf{A} = 8+3 = 11$, $\det \mathbf{B} = 5-0 = 5$ [1]; $\det(\mathbf{AB}) = 80-25 = 55 = 11 \times 5$ ✓ [1]

Method: each entry of $\mathbf{AB}$ is a row of $\mathbf{A}$ times a column of $\mathbf{B}$; for example the top-left entry is $2 \times 1 + (-1) \times (-2) = 4$. For a 2×2 matrix the determinant is $ad - bc$, so $\det \mathbf{A} = 2 \times 4 - (-1) \times 3 = 11$.

The product rule for determinants reflects area scale factors: applying $\mathbf{B}$ then $\mathbf{A}$ multiplies areas by 5 and then by 11. Note that $\mathbf{BA} \neq \mathbf{AB}$ in general, but both have determinant 55.

Answer to Question 2 (4 marks)

a. $k(k \cdot 1 - 1 \cdot 1) - 1(2 \cdot 1 - 1 \cdot 0) + 0$ [1] $= k^2 - k - 2$ ✓ [1]

b. $(k-2)(k+1) = 0$: $k = 2$ **or** $k = -1$ [1]

c. $\det = 9 - 3 - 2 = 4$; volume $= 8 \times 4 =$ **32 cubic units** [1]

Method: expand along the first row, using $\det \begin{pmatrix} k & 1 \\ 1 & 1 \end{pmatrix} = k - 1$ and $\det \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = 2$, with the sign pattern $+ \; - \; +$. A matrix is singular when its determinant is zero, so $\mathbf{N}$ has no inverse for these values of k.

In (c), the determinant is the volume scale factor: the cube has volume $2^3 = 8$, and multiplying by $|\det \mathbf{N}| = 4$ gives 32. Check (b): with $k = 2$, $4 - 2 - 2 = 0$ ✓.

Answer to Question 3 (4 marks)

a. T_1: $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ [1]; T_2: $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ [1]

b. $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ [1]: **reflection in the line** $y = -x$ [1]

Method: the columns of a transformation matrix are the images of $\binom{1}{0}$ and $\binom{0}{1}$. A clockwise rotation of 90° sends $\binom{1}{0}$ to $\binom{0}{-1}$ and $\binom{0}{1}$ to $\binom{1}{0}$. For T_1 followed by T_2, the matrix is $\mathbf{T_2 T_1}$: the first transformation is written on the right.

Check: $(1,0) \mapsto (0,-1) \mapsto (0,-1)$, and $(0,-1)$ is the reflection of $(1,0)$ in $y=-x$ ✓. The determinant is -1, as expected for a reflection.

Answer to Question 4 (4 marks)

a. $\mathbf{M}\begin{pmatrix} t \\ -t \end{pmatrix} = \begin{pmatrix} 2t-t \\ 2t-3t \end{pmatrix} = \begin{pmatrix} t \\ -t \end{pmatrix}$ ✓ [1]
b. $(x, mx) \mapsto (2x+mx,\ 2x+3mx)$; $2+3m = m(2+m)$ [1], so $m^2 - m - 2 = 0$, $m=-1$ or $m=2$: $y=2x$ [1]
c. $(1,2) \mapsto (4,8)$, so points are multiplied by 4: **invariant line, not a line of invariant points** [1]

Method: a line $y=mx$ is invariant if the image of every point on it lies on the same line, which gives an equation for m. A line of invariant points is stronger: each point must map to itself. In (b), the root $m=-1$ recovers the line from (a), a useful check. The eigenvalue for $y=2x$ is 4, so points move away from the origin along that line.

Answer to Question 5 (4 marks)

a. $P'(8,4)$, $Q'(2,6)$ [1]
b. Field: $\frac{1}{2} \times 4 \times 2 = 4$ square units ($40\,000$ m²) [1]; $\det \mathbf{M} = 6 - 1 = 5$, so image area **20 square units** [1]
c. $\mathbf{M}^{-1} = \frac{1}{5}\begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}$, $\frac{1}{5}\begin{pmatrix} 15-5 \\ -5+10 \end{pmatrix}$: $(2,1)$, i.e. 200 m east and 100 m north of O [1]

Method: the determinant of $\mathbf{M}$ is the area scale factor, so the image area is 5×4. In real units the image represents $200\,000$ m², five times the true field. In (c), solve $\mathbf{Mx} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$ by premultiplying by $\mathbf{M}^{-1}$. Check: $\mathbf{M}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4+1 \\ 2+3 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$ ✓.