Lesson - Matrices and linear transformations

Year: Further Maths

Matrices store linear transformations of the plane and of space. This chapter covers matrix algebra, determinants, the standard 2D transformations, composition, and invariant points and lines.

Starter activity

Draw the unit square with corners $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$ on squared paper. Now apply the rule $(x, y) \mapsto (2x + y,\ x + y)$ to each corner and draw the image. What shape do you get, and what is its area? Try the rule $(x, y) \mapsto (-y,\ x)$ as well. Can you see a quick way to predict the image of the whole square from the images of just two corners, and is there a single number that tells you the new area?

Matrix algebra and determinants

Rule: matrix multiplication

Matrices are added element by element. $\mathbf{AB}$ exists when the number of columns of $\mathbf{A}$ equals the number of rows of $\mathbf{B}$, and $(\mathbf{AB})_{ij} = \sum_k a_{ik} b_{kj}$. Multiplication is **associative** but not commutative: in general $\mathbf{AB} \neq \mathbf{BA}$. Also $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$ and $(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$.

Definition: determinant

For $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det \mathbf{A} = ad - bc$. For 3×3, expand along a row with alternating signs $+ - +$:

A matrix is **singular** if $\det = 0$; it has no inverse.

Key fact: rules for determinants

$\det(\mathbf{AB}) = \det \mathbf{A} \det \mathbf{B}$, and $\det(k\mathbf{A}) = k^n \det \mathbf{A}$ for an $n \times n$ matrix.

Example 1

Let $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 4 \\ 1 & -2 \end{pmatrix}$. Find $\mathbf{AB}$ and $\mathbf{BA}$ and check the determinant rule.

$\mathbf{AB} = \begin{pmatrix} 1 & 6 \\ 3 & -10 \end{pmatrix}$ but $\mathbf{BA} = \begin{pmatrix} -4 & 12 \\ 4 & -5 \end{pmatrix}$. The determinants of $\mathbf{A}$ and $\mathbf{B}$ are 7 and -4, and $\det(\mathbf{AB}) = -10 - 18 = -28 = 7 \times (-4)$.

Example 2: 3 by 3 determinant

Evaluate $\begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{vmatrix}$ **.**

$1(0-1) - 0 + 2(0-3) = -1 - 6 = -7$. Expanding along a row containing a zero saves work.

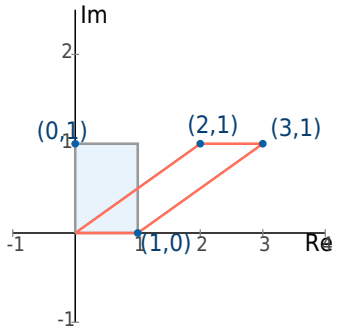
Linear transformations in two dimensions

Definition: matrix of a transformation

The columns of the matrix are the images of the unit vectors $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ sends $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$.

Key fact: standard transformations

Transformation	Matrix
Rotation θ anticlockwise about O	$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$
Reflection in $y = x\tan\theta$	$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$
Enlargement scale factor k, centre O	$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$
One-way stretch	$\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}$
Shear, x-axis invariant	$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$



Shear [[1,2],[0,1]]: the unit square (grey) maps to a parallelogram (coral) of the same area.

Method: finding a matrix from a description

1. Decide where $\begin{pmatrix}1\\0\end{pmatrix}$ goes.
2. Decide where $\begin{pmatrix}0\\1\end{pmatrix}$ goes.
3. Write these two images as the first and second columns.
4. Test the matrix on a point not on the invariant line.

Rule: composition and area

If $\mathbf{P}$ is applied first and then $\mathbf{Q}$, the combined matrix is $\mathbf{QP}$: the first transformation is written *nearest the vector*. The **area scale factor** is $|\det \mathbf{M}|$; if $\det \mathbf{M}$, orientation is reversed.

Example 3

Reflection in the x-axis is followed by an anticlockwise rotation through $90°$. Find the single matrix.

$\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$, a reflection in $y = x$.

Example 4

Find the matrix for a reflection in $y = x$ followed by an enlargement of scale factor 2.

$2\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} = \begin{pmatrix}0 & 2\\2 & 0\end{pmatrix}$. Its determinant is -4, so areas are multiplied by 4 and orientation is reversed.

Watch out

"$\mathbf{P}$ then $\mathbf{Q}$" is $\mathbf{QP}$, not $\mathbf{PQ}$. Also use $|\det|$ for the area scale factor, since $\det$ may be negative.

Invariant points and invariant lines

Definition: invariant point and line

An **invariant point** satisfies $\mathbf{Mx} = \mathbf{x}$, so solve $(\mathbf{M} - \mathbf{I})\mathbf{x} = \mathbf{0}$; the origin always is one. An **invariant line** maps onto itself as a whole, though points may move along it.

Rule: invariant lines and eigenvectors

If $\mathbf{Mv} = \lambda\mathbf{v}$, the line through the origin in the direction $\mathbf{v}$ is invariant, and λ is the scale factor along it. The eigenvalues satisfy $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$.

Example 5

Find the invariant lines through the origin of $\mathbf{M} = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$.

$\lambda^2 - 5\lambda + 6 = 0 \Rightarrow \lambda = 3, 2$. For $\lambda = 3$: $\mathbf{v} = \binom{1}{0}$, the line $y = 0$. For $\lambda = 2$: $x + y = 0$, $\mathbf{v} = \binom{1}{-1}$, the line $y = -x$.

Three dimensions

Note: transformations of space

3×3 matrices represent transformations of space. For example rotates 90° anticlockwise about the z-axis (viewed from above), and $\operatorname{diag}(-1, 1, 1)$ reflects in the plane $x = 0$. The volume scale factor is $|\det \mathbf{M}|$.

Watch out

- Forgetting the signs $+ - +$ in a 3×3 determinant.
- Confusing invariant *points* ($\mathbf{Mx} = \mathbf{x}$) with invariant *lines*.
- When asked to describe a transformation, giving only its type: add the angle and centre, the mirror line, the scale factor, or the invariant line of a shear.

More methods and worked examples

Method: finding invariant lines

1. For lines through the origin, take a general point (x, mx) and find its image (X, Y) using the matrix.
2. Require the image to lie on the same line: $Y = mX$. This gives a quadratic

equation in m.

3. Solve the quadratic. Each real root gives an invariant line $y = mx$. Remember to check for the vertical line $x = 0$ separately, because it cannot be written in the form $y = mx$.
4. For lines not through the origin, use $(x, mx + c)$ and require the image to satisfy $Y = mX + c$ for all x; compare the coefficient of x and the constant term.
5. To decide whether a line is a line of invariant points, check whether $\mathbf{Mx} = \mathbf{x}$ for every point on it, not just whether the line maps to itself.

Worked example 6: composing a stretch and a rotation

Find the single matrix for a stretch parallel to the y-axis with scale factor 3 followed by a rotation of 90° anticlockwise about O, and find the image of the point $(1, 2)$.

The stretch has matrix $\mathbf{S} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ and the rotation has matrix $\mathbf{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The stretch happens first, so it sits next to the vector and the product is $\mathbf{RS} = \begin{pmatrix} 0 & -3 \\ 1 & 0 \end{pmatrix}$. The image of $(1, 2)$ is $(0 - 6,\ 1 + 0) = (-6, 1)$. We can check this step by step: the stretch sends $(1, 2)$ to $(1, 6)$ and the rotation then sends $(1, 6)$ to $(-6, 1)$, which agrees. The determinant of the product is 3, which is the product of the determinants 3 and 1, so areas are tripled.

Worked example 7: area in context

A logo with area 12 cm² is transformed by the matrix $\begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix}$. Find the area of the new logo.

The determinant is $3 \times 1 - (-2) \times 4 = 3 + 8 = 11$. Area is multiplied by the absolute value of the determinant, so the new logo has area $12 \times 11 = 132$ cm². If the determinant had been -11, the area would still be 132 cm², but the logo would appear as a mirror image because the orientation of the shape would be reversed.

Worked example 8: a line of invariant points

Show that the matrix $\begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}$ has a line of invariant points and find it.

Invariant points satisfy $3x - 2y = x$ and $x = y$. The first equation simplifies to $2x = 2y$, which is the same as the second, so every point with $y = x$ is invariant. The line of invariant points is $y = x$. Because the two equations are not independent, the matrix $\mathbf{M} - \mathbf{I} = \begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix}$ has determinant zero, which is always the sign that a line of invariant points exists. Using the method above, the other invariant line through the origin is $y = \frac{1}{2}x$: the image of $(2, 1)$ is $(4, 2)$, so points on it are moved away from the

origin by a scale factor of 2.

Worked example 9: a rotation in three dimensions

Write down the matrix for a rotation of θ about the z-axis and use it to find the image of $(2, 0, 5)$ under a rotation of 60°.

The z-axis is fixed, so $\mathbf{k} \mapsto \mathbf{k}$, and in the xy-plane the rotation acts as in two dimensions. The matrix is  . With $\theta = 60^\circ$ the image of $(2, 0, 5)$ is $(2\cos 60^\circ,\ 2\sin 60^\circ,\ 5) = (1, \sqrt{3}, 5)$. The height is unchanged and the distance from the z-axis stays equal to 2, as we would expect for a rotation about that axis.

Watch out: the order of a product

The most common error in this chapter is writing a composite transformation in the wrong order. If $\mathbf{A}$ is applied first and then $\mathbf{B}$, the single matrix is $\mathbf{BA}$, not $\mathbf{AB}$, because the matrix that acts first must be next to the vector. A useful way to remember this is to read the product from right to left, in the same way as a composite function $f(g(x))$ means "first g, then f". Always check a composite matrix with a simple point, such as $(1, 0)$, by applying the transformations one at a time.

Shears and stretches

A shear keeps one line fixed, called the invariant line of the shear, and moves every other point parallel to that line by a distance proportional to its distance from it. The matrix $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ is a shear with the x-axis fixed, and its determinant is 1, so shears never change area. A stretch parallel to the x-axis with scale factor k has matrix $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$; here the y-axis is the line of invariant points. In an examination, describe a shear by stating the invariant line and the image of one point not on it, and describe a stretch by stating its direction, its scale factor and its invariant line.

Exam technique for transformations

When you are asked to describe a transformation fully, give every piece of information that defines it: a rotation needs a centre, an angle and a direction; a reflection needs the mirror line; an enlargement needs a centre and a scale factor. Use the determinant as a quick check: rotations have determinant 1, reflections have determinant -1, and enlargements with scale factor k have determinant k^2. If a question says "show that", write each multiplication out in full so that the examiner can follow every entry of the product, and finish with a sentence that states clearly what you have shown.

Did you know?

The word matrix was introduced by James Joseph Sylvester in 1850, and his friend Arthur Cayley published the rules of matrix algebra in 1858, including the remarkable result now called the Cayley-Hamilton theorem: every square matrix satisfies its own characteristic equation, as in Question 1 where $\mathbf{A}^2 - 7\mathbf{A} + 14\mathbf{I} = \mathbf{0}$. Today every computer game and animated film uses matrices of exactly the kind in this chapter: each frame, the position of millions of points is rotated, scaled and projected by multiplying by matrices, and graphics cards are designed to do these multiplications very quickly.

Otto's tip

To find a matrix from a geometric description, track where $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ go and write the images as columns. Then check it on one more point.