Worksheet - Matrices and linear transformations

Year: Further Maths · 20

Show your working.

Question 1

Question 1 – Multiplying matrices *(Section A · Skills practice)*

The matrices are $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & 1 \end{pmatrix}$.

a. Find $\mathbf{AB}$ and $\mathbf{BA}$. What does your answer show?
b. Find $\mathbf{AC}$ and explain why $\mathbf{CA}$ does not exist.
c. Find $\mathbf{A}^2$ and show that $\mathbf{A}^2 - 7\mathbf{A} = -14\mathbf{I}$.
d. Hence write $\mathbf{A}^3$ in the form $p\mathbf{A} + q\mathbf{I}$.

Question 2

Question 2 – Determinants *(Section A · Skills practice)*

Evaluate each determinant, showing your expansion.

a.

b. , expanding along the first row.

c. Show that .

d. Hence find the values of k for which this matrix is singular.

Question 3

Question 3 – Standard transformation matrices *(Section A · Skills practice)*

Use the images of $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to write down the matrix of each transformation, then find the image of the point $P(2, -1)$.

a. A rotation of 90° anticlockwise about the origin.
b. A reflection in the line $y = -x$.
c. An enlargement with centre O and scale factor 3.
d. A stretch parallel to the x-axis with scale factor 2.

Question 4

Question 4 – Describing a matrix *(Section A · Skills practice)*

Describe fully the single transformation represented by each matrix. In each case, state the determinant and say what it tells you.

a. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

b. $\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

c. $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

d. $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$

Question 5

Question 5 – Combining two transformations *(Section A · Skills practice)*

$\mathbf{P}$ is a reflection in the x-axis and $\mathbf{Q}$ is a rotation of 90° anticlockwise about the origin.

a. Write down the matrices $\mathbf{P}$ and $\mathbf{Q}$.
b. Find the matrix of the transformation "$\mathbf{P}$ followed by $\mathbf{Q}$" and describe it as a single transformation.
c. Find the matrix of "$\mathbf{Q}$ followed by $\mathbf{P}$" and describe it.
d. Use the point $(3, 1)$ to check your answer to part (b).

Question 6

Question 6 – Invariant points and lines *(Section A · Skills practice)*

The transformation T has matrix $\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$.

a. Show that the origin is the only invariant point of T.
b. By considering the image of the point (x, mx), find the invariant lines of T that pass through the origin.
c. Is either of these lines a line of invariant points? Explain.

Question 7

Question 7 – A photo editing app *(Section B · Real-life practice)*

A photo editing app can tilt a picture with the shear $\mathbf{S} = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$ and widen it with the

stretch $\mathbf{W} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. A rectangular photo has corners $(0, 0)$, $(4, 0)$, $(4, 3)$ and $(0, 3)$, with units in centimetres.

a. Find the corners of the photo after the shear.
b. Explain why the area of the photo is not changed by the shear.
c. The user applies the shear and then the stretch. Find the single matrix for the combined effect.
d. Find the area of the final image and the coordinates of the image of $(4, 3)$.

Question 8

Question 8 – Tiles on a kitchen floor *(Section B · Real-life practice)*

A designer makes a tile pattern by applying the matrix $\mathbf{M} = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$ to a unit square tile with vertices $O(0, 0)$, $A(1, 0)$, $B(1, 1)$ and $C(0, 1)$. One unit on the plan represents 30 cm.

a. Find the images of A, B and C.
b. Find the area of the image tile on the plan.
c. Find the real area of one image tile in cm².
d. The floor has area 9.45 m². Ignoring cutting, how many image tiles are needed to cover it?

Question 9

Question 9 – A robot arm *(Section B · Real-life practice)*

A robot arm is fixed at the origin and holds a tool at $(10, 0)$, in centimetres. The controller rotates the arm by 30° anticlockwise and then by a further 45° anticlockwise.

a. Write down exact matrices $\mathbf{R}_{30}$ and $\mathbf{R}_{45}$ for the two rotations.
b. Find the product $\mathbf{R}_{45}\mathbf{R}_{30}$ and show that it is the matrix of a rotation of 75°.
c. Hence show that $\cos 75^\circ = \dfrac{\sqrt{6} - \sqrt{2}}{4}$.
d. Find the final position of the tool, to 3 significant figures.
e. Does the order of the two rotations matter? Explain.

Question 10

Question 10 – Redrawing a garden plan *(Section B · Real-life practice)*

A triangular flower bed on a garden plan has vertices $(0, 0)$, $(6, 0)$ and $(0, 4)$, measured in metres. To design a matching bed on the other side of a path, the gardener applies an enlargement with centre O and scale factor 2, followed by a reflection in the y-axis.

a. Write down the matrices of the two transformations.

b. Find the single matrix for the combined transformation.
c. Find the vertices of the new bed.
d. Find the area of the original bed and use the determinant to find the area of the new bed.

Question 11

Question 11 – Mirror in a video game *(Section B · Real-life practice)*

In a video game, a mirror is placed along the line $y = x\tan 30^\circ$ through the origin, so that a character is reflected in this line.

a. Use the formula $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$ to write down the exact matrix $\mathbf{R}$ of the reflection.
b. Find the image of the point $(2, 0)$.
c. Show that the point $(\sqrt{3}, 1)$ is invariant and explain why.
d. Show that $\mathbf{R}^2 = \mathbf{I}$ and interpret this in the game.

Question 12

Question 12 – Flattening a design *(Section B · Real-life practice)*

A printing machine is faulty: instead of copying a design, it applies the matrix $\mathbf{F} = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$ to every point.

a. Find $\det \mathbf{F}$ and explain what this tells you about the area of the printed image.
b. Show that the image of every point lies on the line $y = 2x$.
c. Find all the points that are printed at the origin.
d. Explain why the original design cannot be recovered from the printed image.

Question 13

Question 13 – A rotating stage *(Section B · Real-life practice)*

A theatre stage can rotate, and a lighting rig above it can tilt. Take axes with z vertical. The stage rotation is 90° about the z-axis and the tilt is 90° about the x-axis, both anticlockwise when viewed from the positive axis.

a. Write down the 3×3 matrices $\mathbf{R}_z$ and $\mathbf{R}_x$.
b. A prop is at $(1, 2, 3)$. Find its image under $\mathbf{R}_z$.
c. Find the image of $(1, 2, 3)$ under $\mathbf{R}_z$ followed by $\mathbf{R}_x$.
d. Find the image under $\mathbf{R}_x$ followed by $\mathbf{R}_z$, and comment.

Question 14

Question 14 – Packing boxes *(Section B · Real-life practice)*

A packaging company models the distortion of a box in transport by the matrix

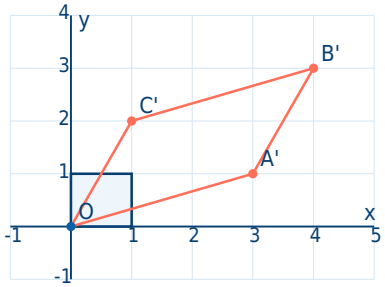
, applied to the coordinates of the corners (in cm).

a. Find the image of the corner $(1, 1, 1)$.
b. Calculate $\det \mathbf{D}$, expanding along the first row.
c. A box measures 2 cm by 3 cm by 4 cm before transport. Find the volume of its image.
d. Explain how the determinant would show that a transformation flattens the box completely.

Question 15

Question 15 – Reading a transformation from a diagram *(Section C · Problem solving)*

The diagram shows the unit square $OABC$, with $A(1, 0)$, $B(1, 1)$ and $C(0, 1)$, and its image $OA'B'C'$ under a linear transformation T. The image has vertices $A'(3, 1)$, $B'(4, 3)$ and $C'(1, 2)$.



a. Write down the matrix $\mathbf{M}$ of T, explaining your reasoning.
b. Find the area of the parallelogram $OA'B'C'$.
c. Find the point that T maps to $(7, 4)$.
d. Find the gradients of the invariant lines of T through the origin, giving exact answers.
e. A student says that T is a rotation followed by an enlargement. Use the determinant and the images of $\mathbf{i}$ and $\mathbf{j}$ to show that the student is wrong.

Question 16

Question 16 – A mirror line y = 2x *(Section C · Problem solving)*

The matrix $\mathbf{R} = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}$ is given. A student records some points and their images in a table.

Point	$(5,0)$	$(0,5)$	$(3,1)$	$(1,2)$
Image	$(-3,4)$	$(4,3)$	$(\ldots,\ldots)$	$(\ldots,\ldots)$

a. Complete the table.
b. Show that every point on the line $y = 2x$ is invariant.
c. Show that the line $y = -\frac{1}{2}x$ is invariant but is not a line of invariant points.
d. Find $\det \mathbf{R}$ and $\mathbf{R}^2$, and hence describe the transformation fully.
e. Write $\cos 2\theta$ and $\sin 2\theta$ in terms of $\tan\theta = 2$ and use them to confirm your description with the general reflection matrix.

Question 17

Question 17 – Transforming a logo *(Section C · Problem solving)*

A designer transforms a logo of area 6 cm² by applying three transformations in the order shown in the table.

Step	Transformation	Matrix
1	Enlargement, centre O, scale factor 2	$\mathbf{E}$
2	Rotation 90° anticlockwise about O	$\mathbf{R}$
3	Reflection in $y = x$	$\mathbf{F}$

a. Write down the matrices $\mathbf{E}$, $\mathbf{R}$ and $\mathbf{F}$.
b. Explain why the single matrix for the whole process is $\mathbf{FRE}$, and calculate it.
c. Describe the combined transformation as simply as possible.
d. Find the area of the final logo.
e. A point of the logo ends at $(6,-2)$. Find where it started.

Question 18

Question 18 – Reflections in space *(Section C · Problem solving)*

In three dimensions, $\mathbf{A}$ is the reflection in the plane $x = y$ and $\mathbf{B}$ is the reflection in the plane $z = 0$.

a. Find the images of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ under each reflection, and hence write down $\mathbf{A}$ and $\mathbf{B}$.
b. Find the matrix $\mathbf{C}$ of $\mathbf{A}$ followed by $\mathbf{B}$.
c. Find $\det \mathbf{C}$ and explain what it tells you about volume and orientation.
d. Find the invariant points of $\mathbf{C}$ and describe them geometrically.
e. Find $\mathbf{C}^2$ and hence describe $\mathbf{C}$ as a single transformation.

Question 19

Question 19 – Repeating a shear *(Section C · Problem solving)*

The shear $\mathbf{S} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is applied repeatedly to the point $P(2, 3)$, which models a deck of cards being pushed sideways one step at a time.

n	1	2	3	4
$\mathbf{S}^n$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\dots$	$\dots$	$\dots$

a. Calculate $\mathbf{S}^2$, $\mathbf{S}^3$ and $\mathbf{S}^4$ and complete the table.
b. Suggest a formula for $\mathbf{S}^n$, and explain it geometrically.
c. Find the position of P after 10 applications of the shear.
d. Write down $\mathbf{S}^{-10}$ and check that it maps your answer to part (c) back to P.
e. Explain why the area of any shape is unchanged however many times the shear is applied.

Question 20

Question 20 – Challenge: two mirrors make a rotation *(Section D · Challenge)*

A kaleidoscope uses two mirrors through the origin. The matrix of the reflection in the line through O at angle θ to the x-axis is $\mathbf{M}_\theta = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$.

a. Show that $\mathbf{M}_\beta \mathbf{M}_\alpha$ is the matrix of a rotation, and find the angle of the rotation.
b. Hence explain why reflecting in two mirrors at 30° to each other produces a rotation of 60°.
c. Find all matrices $\mathbf{X} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ that commute with the shear $\mathbf{S} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.
d. Show that a reflection never commutes with this shear.