Answers - Matrices and linear transformations

Year: Further Maths

Answer to Question 1

a. Row by column: $\mathbf{AB} = \begin{pmatrix} 3\cdot 1+(-1)\cdot 0 & 3\cdot 2+(-1)(-3) \\ 2\cdot 1+4\cdot 0 & 2\cdot 2+4(-3) \end{pmatrix} = \begin{pmatrix} 3 & 9 \\ 2 & -8 \end{pmatrix}$.
Similarly $\mathbf{BA} = \begin{pmatrix} 3+4 & -1+8 \\ 0-6 & 0-12 \end{pmatrix} = \begin{pmatrix} 7 & 7 \\ -6 & -12 \end{pmatrix}$. The two products are different, so **matrix multiplication is not commutative**.

b. $\mathbf{A}$ is 2×2 and $\mathbf{C}$ is 2×3, so $\mathbf{AC}$ is 2×3: first row $(3-3,\ 0+1,\ 6-1)$, second row $(2+12,\ 0-4,\ 4+4)$, giving $\begin{pmatrix} 0 & 1 & 5 \\ 14 & -4 & 8 \end{pmatrix}$. For $\mathbf{CA}$ the number of columns of $\mathbf{C}$ (3) does not equal the number of rows of $\mathbf{A}$ (2), so the product is **not defined**.

c. $\mathbf{A}^2 = \begin{pmatrix} 9-2 & -3-4 \\ 6+8 & -2+16 \end{pmatrix} = \begin{pmatrix} 7 & -7 \\ 14 & 14 \end{pmatrix}$. Then
$\mathbf{A}^2 - 7\mathbf{A} = \begin{pmatrix} 7 & -7 \\ 14 & 14 \end{pmatrix} - \begin{pmatrix} 21 & -7 \\ 14 & 28 \end{pmatrix} = \begin{pmatrix} -14 & 0 \\ 0 & -14 \end{pmatrix} = -14\mathbf{I}$ ✓

d. Multiply $\mathbf{A}^2 = 7\mathbf{A} - 14\mathbf{I}$ by $\mathbf{A}$: $\mathbf{A}^3 = 7\mathbf{A}^2 - 14\mathbf{A} = 7(7\mathbf{A} - 14\mathbf{I}) - 14\mathbf{A} = 35\mathbf{A} - 98\mathbf{I}$. Check: both give $\begin{pmatrix} 7 & -35 \\ 70 & 42 \end{pmatrix}$.

Answer to Question 2

a. $5\times 4 - (-2)\times 3 = 20 + 6 = \mathbf{26}$

b. **24**

c. $1(2k-1) - 2(0-3) + k(0-3k) = 2k - 1 + 6 - 3k^2 = -3k^2 + 2k + 5$ ✓

d. Singular when $3k^2 - 2k - 5 = 0$, i.e. $(3k-5)(k+1) = 0$, so $k = -1$ **or** $k = \frac{5}{3}$.

Answer to Question 3

a. $\mathbf{i} \mapsto \binom{0}{1}$, $\mathbf{j} \mapsto \binom{-1}{0}$, so the matrix is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. $P \mapsto \binom{0\cdot 2+(-1)(-1)}{1\cdot 2+0} = (1, 2)$.

b. $\mathbf{i} \mapsto \binom{0}{-1}$ and $\mathbf{j} \mapsto \binom{-1}{0}$: $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. $P \mapsto \binom{0+1}{-2+0} = (1, -2)$.

c. $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$; $P \mapsto (6, -3)$.

d. $\mathbf{i} \mapsto \binom{2}{0}$, $\mathbf{j}$ unchanged: $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$; $P \mapsto (4, -1)$ (only the x-coordinate is doubled).

Answer to Question 4

a. $\mathbf{i} \mapsto \mathbf{j}$ and $\mathbf{j} \mapsto \mathbf{i}$: **reflection in the line** $y = x$. Determinant $0 - 1 = -1$: area is unchanged and orientation is reversed.

b. The matrix has the form $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ with $\cos\theta = \frac{1}{2}$, $\sin\theta = \frac{\sqrt{3}}{2}$, so $\theta = 60°$: **rotation of** $60°$ **anticlockwise about** O. Determinant $\frac{1}{4} + \frac{3}{4} = 1$.

c. $\mathbf{i}$ is fixed and $\mathbf{j} \mapsto \binom{3}{1}$: points move parallel to the x-axis by 3 times their y-coordinate. **Shear with the** x**-axis invariant,** $(0, 1) \mapsto (3, 1)$. Determinant 1: area unchanged.

d. Every vector is multiplied by -2: **enlargement with centre** O **and scale factor** -2 (equivalently a rotation of $180°$ followed by an enlargement of scale factor 2). Determinant 4: areas are multiplied by 4.

Answer to Question 5

a. $\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

b. "$\mathbf{P}$ then $\mathbf{Q}$" is the product $\mathbf{QP}$ (the first transformation is on the right): $\mathbf{QP} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, a **reflection in** $y = x$.

c. $\mathbf{PQ} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, a **reflection in** $y = -x$. The order matters.

d.  , and the reflection in $y = x$ sends $(3, 1)$ to $(1, 3)$ ✓

Answer to Question 6

a. Invariant points satisfy $\mathbf{Mx} = \mathbf{x}$, i.e. $3x + y = 0$ and $2x + 2y = 0$. The matrix $\mathbf{M} - \mathbf{I} = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$ has determinant $6 - 2 = 4 \neq 0$, so the **only solution is** $x = y = 0$.

b. $(x, mx) \mapsto (4x + mx,\ 2x + 3mx)$. For the image to lie on $y = mx$: $2x + 3mx = m(4x + mx)$, so $2 + 3m = 4m + m^2$, i.e. $m^2 + m - 2 = 0$, $(m + 2)(m - 1) = 0$. **The invariant lines are** $y = x$ **and** $y = -2x$.

c. No. On $y = x$, $(1, 1) \mapsto (5, 5)$, so points move along the line (scale factor 5). On $y = -2x$, $(1, -2) \mapsto (2, -4)$ (scale factor 2). Each line maps to itself but **its points are not fixed**.

Answer to Question 7

a. Each point $(x, y) \mapsto (x + \frac{1}{2}y,\ y)$: $(0,0)$, $(4,0)$, $(5.5,3)$, $(1.5,3)$.
b. $\det \mathbf{S} = 1 \times 1 - \frac{1}{2} \times 0 = 1$, so the area scale factor is 1: the area stays **12 cm²** (a parallelogram with the same base 4 and height 3).
c. Shear first, so the stretch goes on the left: $\mathbf{WS} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$.
d. $\det(\mathbf{WS}) = 2$, so the area is $12 \times 2 =$ **24 cm²**. $(4,3) \mapsto (2 \times 4 + 3,\ 3) = (11,3)$.

Answer to Question 8

a. $A \mapsto \binom{3}{1}$, $B \mapsto \binom{3-1}{1+2} = \binom{2}{3}$, $C \mapsto \binom{-1}{2}$: $A'(3,1)$, $B'(2,3)$, $C'(-1,2)$.
b. Area factor $= \det \mathbf{M} = 3 \times 2 - (-1) \times 1 =$ **7**, so the image tile has area 7 square units.
c. One square unit is $30 \times 30 = 900$ cm², so the area is $7 \times 900 =$ **6300 cm²**.
d. 9.45 m² $= 94\,500$ cm². $94\,500 \div 6300 =$ **15 tiles**.

Answer to Question 9

a. $\mathbf{R}_{30} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$, $\mathbf{R}_{45} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$
b. Top-left entry: $\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$; bottom-left: $\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$. The product is $\begin{pmatrix} \frac{\sqrt{6}-\sqrt{2}}{4} & -\frac{\sqrt{6}+\sqrt{2}}{4} \\ \frac{\sqrt{6}+\sqrt{2}}{4} & \frac{\sqrt{6}-\sqrt{2}}{4} \end{pmatrix}$, which has the rotation pattern with $\cos\theta$ on the diagonal: **a rotation of** $30^\circ + 45^\circ = 75^\circ$.
c. A rotation of 75° has top-left entry $\cos 75^\circ$, and comparing with part (b) gives $\cos 75^\circ = \dfrac{\sqrt{6}-\sqrt{2}}{4}$ ✓
d. The tool moves to $(10\cos 75^\circ,\ 10\sin 75^\circ) = (2.59,\ 9.66)$ cm.
e. **No**: rotations about the same centre commute, since both orders give a rotation of 75° ($\mathbf{R}_{30}\mathbf{R}_{45} = \mathbf{R}_{45}\mathbf{R}_{30}$).

Answer to Question 10

a. Enlargement $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$; reflection in the y-axis $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.
b. Enlargement first, so it goes on the right: $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$.
c. $(0,0) \mapsto (0,0)$, $(6,0) \mapsto (-12,0)$, $(0,4) \mapsto (0,8)$: $(0,0)$, $(-12,0)$, $(0,8)$.

d. Original area $\frac{1}{2} \times 6 \times 4 = 12$ m². $|\det| = |-4| = 4$, so the new bed has area **48 m²** (check: $\frac{1}{2} \times 12 \times 8 = 48$ ✓). The negative determinant shows that the orientation is reversed by the reflection.

Answer to Question 11

a. $2\theta = 60°$: $\mathbf{R} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

b. $\mathbf{R}\binom{2}{0} = \binom{1}{\sqrt{3}}$: $(1, \sqrt{3})$. (Both points are 2 units from O, at angles $0°$ and $60°$, symmetric about the $30°$ line.)

c. $\mathbf{R}\binom{\sqrt{3}}{1} = \binom{\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}}{\frac{3}{2}-\frac{1}{2}} = \binom{\sqrt{3}}{1}$ ✓. The point lies on the mirror line, since $1 = \sqrt{3}\tan 30°$, and **points on the mirror do not move**.

d. $\mathbf{R}^2 = \begin{pmatrix} \frac{1}{4}+\frac{3}{4} & \frac{\sqrt{3}}{4}-\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4}-\frac{\sqrt{3}}{4} & \frac{3}{4}+\frac{1}{4} \end{pmatrix} = \mathbf{I}$ ✓. **Reflecting twice brings the character back to its starting position**, so the reflection is its own inverse.

Answer to Question 12

a. $\det \mathbf{F} = 2 \times 2 - 1 \times 4 = \mathbf{0}$, so every area is multiplied by 0: the printed image has **zero area** (it is squashed onto a line).

b. $(x, y) \mapsto (2x + y,\ 4x + 2y) = (2x + y,\ 2(2x + y))$, so the image (X, Y) always satisfies $Y = 2X$.

c. $2x + y = 0$: **every point on the line** $y = -2x$ is printed at the origin, for example $(1, -2) \mapsto (0, 0)$.

d. The matrix is **singular** (no inverse): many points share the same image, for example every point on $y = -2x$ goes to O, so the image does not tell us which point it came from.

Answer to Question 13

a. , (the columns are the images of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$).

b. $(-2, 1, 3)$; the height $z = 3$ does not change.

c. : $(-2, -3, 1)$.

d. $\mathbf{R}_x(1,2,3) = (1,-3,2)$, then $\mathbf{R}_z(1,-3,2) = (3,1,2)$. The answers are different: **rotations about different axes do not commute**.

Answer to Question 14

a. $(3,4,2)$

b. $2(3\cdot 1 - 1\cdot 0) - 1(0\cdot 1 - 1\cdot 1) + 0 = 6 + 1 = \mathbf{7}$

c. The volume scale factor is $|\det \mathbf{D}| = 7$. Original volume $2\times 3\times 4 = 24$ cm³, so the image has volume **168 cm³**.

d. A transformation that flattens the box onto a plane or a line gives zero volume, so its **determinant would be 0** (the matrix would be singular).

Answer to Question 15

a. The columns are the images of $\mathbf{i}$ and $\mathbf{j}$, which are A' and C': $\mathbf{M} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$.
Check: $\mathbf{M}\binom{1}{1} = \binom{4}{3} = B'$ ✓

b. $\det \mathbf{M} = 6 - 1 = 5$, and the unit square has area 1, so the area is **5 square units**.

c. $\mathbf{M}^{-1} = \frac{1}{5}\begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}$, so $\mathbf{x} = \frac{1}{5}\binom{14-4}{-7+12} = \frac{1}{5}\binom{10}{5}$: $(2,1)$. Check: $\mathbf{M}\binom{2}{1} = \binom{7}{4}$ ✓

d. $(x, mx) \mapsto (3x + mx,\ x + 2mx)$. Need $1 + 2m = m(3 + m)$, so $m^2 + m - 1 = 0$ and $m = \dfrac{-1 \pm \sqrt{5}}{2}$.

e. A rotation followed by an enlargement keeps angles, so $\mathbf{i}$ and $\mathbf{j}$ would map to perpendicular vectors of equal length. Here $\binom{3}{1}\cdot\binom{1}{2} = 5 \neq 0$ and the lengths $\sqrt{10}$ and $\sqrt{5}$ differ, so T **is not a rotation with an enlargement**.

Answer to Question 16

a. $\mathbf{R}\binom{3}{1} = \frac{1}{5}\binom{-9+4}{12+3} = (-1,3)$; $\mathbf{R}\binom{1}{2} = \frac{1}{5}\binom{-3+8}{4+6} = (1,2)$.

b. $\mathbf{R}\binom{t}{2t} = \frac{1}{5}\binom{-3t+8t}{4t+6t} = \binom{t}{2t}$ ✓ for every t, so $y = 2x$ **is a line of invariant points**.

c. $\mathbf{R}\binom{2t}{-t} = \frac{1}{5}\binom{-6t-4t}{8t-3t} = \binom{-2t}{t}$, which lies on $y = -\frac{1}{2}x$, so the line is invariant. But $(2,-1) \mapsto (-2,1)$, so **points on it move** (they are reversed through O).

d. $\det \mathbf{R} = \frac{1}{25}(-9-16) = -1$ and $\mathbf{R}^2 = \frac{1}{25}\begin{pmatrix} 9+16 & -12+12 \\ -12+12 & 16+9 \end{pmatrix} = \mathbf{I}$. With a line

of invariant points and determinant -1: **reflection in the line** $y = 2x$.

e. $\cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta} = \dfrac{1-4}{5} = -\dfrac{3}{5}$ and $\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta} = \dfrac{4}{5}$. The reflection matrix $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$ is exactly $\mathbf{R}$ ✓

Answer to Question 17

a. $\mathbf{E} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $\mathbf{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\mathbf{F} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

b. The first transformation acts first on the vector, so it is written next to the vector, on the right: $\mathbf{F}(\mathbf{R}(\mathbf{E}\mathbf{x})) = (\mathbf{FRE})\mathbf{x}$. $\mathbf{FR} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, so $\mathbf{FRE} = 2\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$.

c. $(x, y) \mapsto (2x, -2y)$: **a reflection in the x-axis combined with an enlargement, centre O, scale factor 2** (in either order).

d. $|\det| = |-4| = 4$, so the area is $6 \times 4 = $ **24 cm²**.

e. Solve $(2x, -2y) = (6, -2)$: $(3, 1)$.

Answer to Question 18

a. Reflection in $x = y$ swaps the x- and y-coordinates: $\mathbf{i} \leftrightarrow \mathbf{j}$, $\mathbf{k}$ fixed, so

. Reflection in $z = 0$ changes the sign of z: .

b. $\mathbf{C} = \mathbf{BA} =$ [redacted]

c. $\det \mathbf{C} = \det \mathbf{B} \det \mathbf{A} = (-1)(-1) = \mathbf{1}$: volumes are unchanged and orientation is preserved (two reflections).

d. $\mathbf{Cx} = \mathbf{x}$ gives $y = x$, $x = y$, $-z = z$, so $z = 0$ and $x = y$: the invariant points are $(t, t, 0)$**, the line through O with direction $\mathbf{i} + \mathbf{j}$.**

e. $\mathbf{C}^2 = \mathbf{I}$. A transformation with determinant 1, a line of invariant points and $\mathbf{C}^2 = \mathbf{I}$ is **a rotation of $180°$ about the line** $x = y,\ z = 0$.

Answer to Question 19

a. $\mathbf{S}^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $\mathbf{S}^3 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$, $\mathbf{S}^4 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$ (each time the top-right entry increases by 1).

b. $\mathbf{S}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$: each shear moves a point y units to the right, so n shears move it ny units. (This can be proved by induction.)
c. $\mathbf{S}^{10}\binom{2}{3} = \binom{2+30}{3}$: $(32, 3)$.
d. $\mathbf{S}^{-10} = \begin{pmatrix} 1 & -10 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & -10 \\ 0 & 1 \end{pmatrix}\binom{32}{3} = \binom{32-30}{3} = \binom{2}{3}$ ✓
e. $\det \mathbf{S}^n = (\det \mathbf{S})^n = 1^n = 1$, so **every area is multiplied by 1**.

Answer to Question 20

a. $\mathbf{M}_\beta \mathbf{M}_\alpha$ has top-left entry $\cos 2\beta \cos 2\alpha + \sin 2\beta \sin 2\alpha = \cos(2\beta - 2\alpha)$ and bottom-left entry $\sin 2\beta \cos 2\alpha - \cos 2\beta \sin 2\alpha = \sin(2\beta - 2\alpha)$. Similarly the second column is $\binom{-\sin(2\beta-2\alpha)}{\cos(2\beta-2\alpha)}$. So the product is **a rotation through** $2(\beta - \alpha)$ about O.
b. With $\beta - \alpha = 30^\circ$ the angle is $2 \times 30^\circ = 60^\circ$. A kaleidoscope with mirrors at 30° therefore shows copies rotated by multiples of 60°, giving 6-fold symmetry.
c. $\mathbf{XS} = \begin{pmatrix} p & p+q \\ r & r+s \end{pmatrix}$ and $\mathbf{SX} = \begin{pmatrix} p+r & q+s \\ r & s \end{pmatrix}$. Equating: $p = p + r$ gives $r = 0$; $p + q = q + s$ gives $s = p$. So $\mathbf{X} = \begin{pmatrix} p & q \\ 0 & p \end{pmatrix}$ for any p, q.
d. Every matrix that commutes with $\mathbf{S}$ has determinant $p^2 \geq 0$, but a reflection has determinant -1. So **no reflection commutes with** $\mathbf{S}$.