Test answers - Inverse matrices and systems of equations

Year: Further Maths · Duration: 60 minutes · /20

Answer to Question 1 (4 marks)

a. $\det = 8 - 3 = 5$ [1]; $\mathbf{A}^{-1} = \frac{1}{5}\begin{pmatrix} 4 & -3 \\ -1 & 2 \end{pmatrix}$ [1]

b. $\mathbf{x} = \frac{1}{5}\begin{pmatrix} 4 & -3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ 0 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 20 \\ -5 \end{pmatrix}$ [1], so $x = 4$, $y = -1$ [1]

Method: for $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, swap a and d, change the signs of b and c, and divide by the determinant $ad - bc$. Write the system as $\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$ and premultiply by $\mathbf{A}^{-1}$; the order matters because matrix multiplication is not commutative.

Check: $2 \times 4 + 3 \times (-1) = 5$ ✓ and $4 + 4 \times (-1) = 0$ ✓. A quick test of the inverse is $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$.

Answer to Question 2 (5 marks)

a. $\det = 1(1 - 0) - 0 + 2(6 - 0) = 1 + 12 =$ **13** [1]

b. Cofactors: [redacted] [1 for method, 1 for accuracy]. Transpose gives the adjugate; [1]

c. , so $\mathbf{x} = (1, 2, 3)$ [1]

Method: find the matrix of minors, apply the sign pattern to get the cofactors, transpose to obtain the adjugate, then divide by the determinant. In (a), expand along the first row; the zero in position $(1, 2)$ saves one calculation.

Check (c): $1 + 0 + 2 \times 3 = 7$, $2 \times 1 + 2 = 4$ and $3 \times 2 + 3 = 9$ ✓. Forgetting to transpose the cofactor matrix is the most frequent error and loses the accuracy mark.

Answer to Question 3 (4 marks)

a. The third row of the coefficient matrix equals the sum of the first two, so $\det = 0$

[1]

b. The right-hand sides must satisfy $2 + 5 = k$: $k = 7$ [1]
c. Subtract the first two equations: $x + 2z = 3$, so $x = 3 - 2z$ [1]. Then $y = 2 - x - z = z - 1$. With $z = t$: $(x, y, z) = (3 - 2t,\ t - 1,\ t)$ [1]

Geometrically, the three planes form a sheaf when $k = 7$: they share a common line. For any other value of k they form a triangular prism with no common point. Check the general solution in the third equation: $3(3 - 2t) + 2(t - 1) + 4t = 7$ ✓.

Answer to Question 4 (4 marks)

a. The normal of Π_2 is twice the normal of Π_1 [1]. But Π_1 is equivalent to $2x + 4y - 2z = 2$, not 5, so the planes are distinct and parallel [1].
b. Any common point would lie on both Π_1 and Π_2, which do not meet [1]. So there is **no solution**: two parallel planes are cut by the third plane in two parallel lines [1].

Answer to Question 5 (3 marks)

a. $\begin{pmatrix} 3 & 2 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 34 \\ 52 \end{pmatrix}$, $\det = 11$, inverse $\frac{1}{11} \begin{pmatrix} 5 & -2 \\ -2 & 3 \end{pmatrix}$ [2]
b. $\frac{1}{11} \begin{pmatrix} 170 - 104 \\ -68 + 156 \end{pmatrix} = \frac{1}{11} \begin{pmatrix} 66 \\ 88 \end{pmatrix}$: **A costs £6 and B costs £8** [1]

Method: let a and b be the prices in pounds of one bag of A and one bag of B. The coefficient matrix has determinant $3 \times 5 - 2 \times 2 = 11 \neq 0$, so it is non-singular and the solution is unique; premultiply both sides by the inverse.

Check: $3 \times 6 + 2 \times 8 = 34$ ✓ and $2 \times 6 + 5 \times 8 = 52$ ✓. The answer must be given in context, with units, to earn the final mark.