Lesson - Inverse matrices and systems of equations

Year: Further Maths

An inverse matrix undoes a transformation. This chapter finds inverses of 2×2 and 3×3 matrices, uses them to solve simultaneous equations, and interprets the solutions as planes.

Starter activity

Two phone plans are advertised. Plan A costs £10 a month plus 5p per minute, and plan B costs £4 a month plus 8p per minute. Write two equations for the monthly cost C in terms of the number of minutes m, and solve them to find when the plans cost the same. Now imagine that there are three plans and three unknown quantities. Would you still be happy solving by substitution? Think about how a matrix could store the numbers in the equations, and about what operation on the matrix would give you the answer in a single step.

Inverse of a 2 by 2 matrix

Definition: inverse and singular matrices

The **inverse** of $\mathbf{A}$ satisfies $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$. A matrix with $\det \mathbf{A} = 0$ is **singular**: it squashes the plane onto a line (or a point), so cannot be undone.

Rule: 2 by 2 inverse

For $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\det \mathbf{A} = ad - bc \neq 0$:

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Example 1

Find the inverse of $\mathbf{A} = \begin{pmatrix} 2 & -3 \\ 4 & 1 \end{pmatrix}$.

$\det \mathbf{A} = 2 + 12 = 14$, so $\mathbf{A}^{-1} = \frac{1}{14} \begin{pmatrix} 1 & 3 \\ -4 & 2 \end{pmatrix}$.

Key fact: properties of inverses

- $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$: the order reverses (undo your shoes before your socks).

- $\det(\mathbf{A}^{-1}) = \dfrac{1}{\det \mathbf{A}}$ and $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.
- If $\mathbf{AX} = \mathbf{B}$ then $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$; if $\mathbf{XA} = \mathbf{B}$ then $\mathbf{X} = \mathbf{B}\mathbf{A}^{-1}$. The side matters.
- Cayley-Hamilton for 2×2: $\mathbf{A}^2 - (\operatorname{tr}\mathbf{A})\mathbf{A} + (\det \mathbf{A})\mathbf{I} = \mathbf{0}$, which gives $\mathbf{A}^{-1}$ quickly.

Inverse of a 3 by 3 matrix

Method: inverse by cofactors

1. Find $\det \mathbf{A}$. If it is 0, stop: the matrix is singular.
2. Form the matrix of **minors** (the 2×2 determinant left when row i and column j are deleted).
3. Apply the alternating signs  to get the cofactors.
4. Transpose to get the **adjugate**, then $\mathbf{A}^{-1} = \dfrac{1}{\det \mathbf{A}} \operatorname{adj} \mathbf{A}$.
5. Check that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$ for at least one row.

Watch out

Do not forget to transpose the cofactor matrix, and do not forget the factor $\dfrac{1}{\det \mathbf{A}}$ in front. Show the method in non-calculator questions; a calculator may be used only to check.

Solving simultaneous equations

Rule: matrix form of a system

Write the system as $\mathbf{Ax} = \mathbf{b}$. If $\det \mathbf{A} \neq 0$ there is a **unique solution** $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. If $\det \mathbf{A} = 0$ there is either no solution (inconsistent) or infinitely many (consistent).

Example 2

Solve $x + 2y = 4$ and $3x + 7y = 13$.

The matrix $\begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}$ has determinant $7 - 6 = 1$ and inverse $\begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix}$. Then $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 13 \end{pmatrix} = \begin{pmatrix} 28-26 \\ -12+13 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Check: $2 + 2 = 4$ and $6 + 7 = 13$.

Method: a singular system

1. Find the value of the parameter that makes $\det \mathbf{A} = 0$.
2. Combine the equations by elimination.
3. If you reach $0 = 0$, there are infinitely many solutions: introduce a parameter t.
4. If you reach $0 = \text{non-zero}$, there is no solution.

Example 3

Discuss the solutions of $x + y + z = 2$, $x + 2y + 3z = 5$, $2x + 3y + 4z = k$.

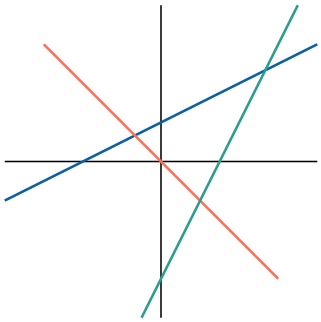
Adding the first two equations gives $2x + 3y + 4z = 7$. So there is no solution unless $k = 7$. For $k = 7$, put $z = t$: $y = 3 - 2t$ and $x = t - 1$, a line of solutions.

Geometrical interpretation

Key fact: three planes

Each equation $ax + by + cz = d$ is a plane. The possible arrangements of three planes are:

Case	Algebra	Picture
Unique solution	$\det \neq 0$	planes meet at a point
Line of solutions	$\det = 0$, consistent, rank 2	planes share a common line (a sheaf)
Plane of solutions	rank 1, consistent	all three planes coincide
No solution	$\det = 0$, inconsistent	triangular prism, or two or three parallel planes



Three lines in 2D forming a triangle: the analogue of three planes forming a triangular prism, meaning no common point.

Example 4

For $x + y + z = 1$, $x + 2y + 3z = 2$, $2x + 3y + 4z = k$, describe the planes.

The left side of the third equation is the sum of the first two, so the system is consistent only when $k = 3$. Then the planes meet in a line. Otherwise they form a

triangular prism and there is no solution.

Watch out

- Using $\mathbf{A}^{-1}\mathbf{B}$ when the equation is $\mathbf{XA} = \mathbf{B}$.
- Concluding "no solution" just because $\det = 0$; there may be infinitely many.

Note: when does an inverse exist?

A square matrix has an inverse exactly when its determinant is not zero. For a 2×2 matrix this means that the two columns are not parallel, so the transformation does not squash the plane onto a line. For a 3×3 matrix it means that the three columns do not lie in a common plane, so the transformation does not flatten space. When the determinant is zero the matrix is called singular, and information has been lost: many different points are sent to the same image, so there is no way of recovering the original. This is the reason why a singular matrix has no inverse, and also why a system of equations with a singular coefficient matrix either has no solutions or infinitely many.

Method: finding the inverse of a 3 by 3 matrix

1. Calculate the determinant by expanding along any row or column. If it is zero, stop: there is no inverse.
2. Find the matrix of minors: for each entry, delete its row and column and find the determinant of the remaining 2×2 matrix.
3. Apply the chessboard pattern of signs  to obtain the matrix of cofactors.
4. Transpose the cofactor matrix to get the adjugate.
5. Divide the adjugate by the determinant, and check that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$.

Worked example 5: the inverse of a product

Given $\mathbf{P} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, find $(\mathbf{PQ})^{-1}$ and verify it.

The inverses are $\mathbf{P}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ because $\det \mathbf{P} = 1$, and $\mathbf{Q}^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$ because $\det \mathbf{Q} = 1$. The rule for the inverse of a product reverses the order, so $(\mathbf{PQ})^{-1} = \mathbf{Q}^{-1}\mathbf{P}^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$. To verify, multiply out $\mathbf{PQ} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$, which has determinant 1, so its inverse is $\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$, as found. The reversal of order is easy to understand with a real-life example: if you put on your

socks and then your shoes, to undo this you must take off your shoes first and then your socks.

Key fact: the three possibilities for a system of equations

A system of n linear equations in n unknowns written as $\mathbf{A}\mathbf{x} = \mathbf{b}$ has exactly one of three outcomes. If $\det \mathbf{A} \neq 0$, there is a unique solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. If $\det \mathbf{A} = 0$, the system either has no solution (inconsistent), or infinitely many solutions (consistent but with a free parameter). To distinguish between the last two cases, eliminate variables, and see whether you obtain a contradiction such as $0 = 3$, or an identity such as $0 = 0$. A system is never found with exactly two solutions, because the solution set of a linear system is a point, a line, a plane, or empty.

Worked example 6: three planes with no unique solution

Show that the planes $x + 2y + 3z = 4$, $2x + y + z = 3$ and $4x + 5y + 7z = k$ meet along a line for one value of k, and find that line.

Look for a relationship between the rows: the third row $(4, 5, 7)$ equals $2 \times (1, 2, 3) + (2, 1, 1)$ because $2 + 2 = 4$, $4 + 1 = 5$ and $6 + 1 = 7$. Therefore the third equation is consistent with the first two only if $k = 2 \times 4 + 3 = 11$. For $k = 11$, the system reduces to two equations. Eliminate x: subtract twice the first from the second to get $-3y - 5z = -5$, so $y = \dfrac{5 - 5z}{3}$. Then $x = 4 - 2y - 3z = 4 - \dfrac{10 - 10z}{3} - 3z = \dfrac{2 + z}{3}$. The line is $(x, y, z) = \left(\dfrac{2 + t}{3}, \dfrac{5 - 5t}{3}, t\right)$. For any other value of k the planes form a triangular prism with no common point.

Watch out: common errors

- Applying the sign pattern wrongly when finding cofactors. The sign of the entry in row i, column j is $(-1)^{i+j}$.
- Forgetting to transpose the cofactor matrix. The adjugate is the transpose, and the error will usually give an inverse that does not check.
- Writing $(\mathbf{AB})^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1}$ instead of $\mathbf{B}^{-1}\mathbf{A}^{-1}$.
- Multiplying on the wrong side when solving matrix equations: for $\mathbf{AX} = \mathbf{C}$ use $\mathbf{X} = \mathbf{A}^{-1}\mathbf{C}$, and for $\mathbf{XA} = \mathbf{C}$ use $\mathbf{X} = \mathbf{C}\mathbf{A}^{-1}$.
- Concluding that a system has no solution because the determinant is zero. It might have infinitely many.

Applications of matrix inverses

Whenever several unknown quantities are linked by several linear relationships, a matrix inverse provides the answer. Engineers use them to find the forces in the

members of a bridge or the currents in a network. Economists use them in input-output models, where each industry buys from the others. Computer graphics programmes use inverse matrices to undo a transformation, for example when the mouse pointer is converted from screen coordinates back into the position in a three-dimensional scene. Cryptographers use them to decode messages that were scrambled by a matrix. In each case the structure is the same: a known linear transformation has been applied to an unknown input, and the inverse recovers the input from the output.

Method: deciding the geometry of three planes

1. Write the system as a matrix equation and calculate the determinant of the coefficient matrix.
2. If it is non-zero, the planes meet at a single point, which you can find by using the inverse.
3. If it is zero, try to eliminate one variable and look for a contradiction. A contradiction means no common point: either a triangular prism (no two planes parallel) or two or more parallel planes.
4. If there is no contradiction, the planes share a line (or all three are the same plane); find the line by setting one variable equal to a parameter.
5. Check parallel planes by comparing the normal vectors.

Why the order of multiplication matters

Matrix multiplication is not commutative, which means that $\mathbf{AB}$ and $\mathbf{BA}$ are usually different. This is the reason why you must be careful about the side on which you multiply when solving a matrix equation. If you have the equation $\mathbf{AX} = \mathbf{C}$ and you want to isolate $\mathbf{X}$, you need the product $\mathbf{A}^{-1}\mathbf{A}$ to appear next to $\mathbf{X}$, so you multiply both sides on the left by $\mathbf{A}^{-1}$. If you multiplied on the right you would obtain $\mathbf{AXA}^{-1}$, which does not simplify. In a similar way, for $\mathbf{XA} = \mathbf{C}$ the inverse goes on the right. A good habit is to write the step explicitly, such as "pre-multiply both sides by $\mathbf{A}^{-1}$", so that the examiner can see that you know which side is correct. The same care is needed when expanding products: $(\mathbf{A}+\mathbf{B})^2$ is $\mathbf{A}^2 + \mathbf{AB} + \mathbf{BA} + \mathbf{B}^2$, and you may not combine the two middle terms.

Worked example 7: a system solved by elimination as a check

Solve $2x + y - z = 3$, $x - y + 2z = 1$, $3x + 2y + z = 10$ by row operations and confirm your answer using a determinant.

From the second equation, $x = 1 + y - 2z$. Substituting into the first gives $2 + 2y - 4z + y - z = 3$, which simplifies to $3y - 5z = 1$. Substituting into the third gives $3 + 3y - 6z + 2y + z = 10$, which simplifies to $5y - 5z = 7$. Subtracting the equation $3y - 5z = 1$ from $5y - 5z = 7$ gives $2y = 6$, so $y = 3$. Then $5z = 3y - 1 = 8$, so $z = \frac{8}{5}$, and $x = 1 + 3 - \frac{16}{5} = \frac{4}{5}$. The determinant of the coefficient matrix is $2(-1-4) - 1(1-6) + (-1)(2+3) = -10 + 5 - 5 = -10$, which is not zero, so the

solution is unique, as the calculation suggested. It is worth checking one equation: $2\left(\frac{4}{5}\right)+3-\frac{8}{5}=\frac{8}{5}+3-\frac{8}{5}=3$, which is correct.

Reading the geometry from the algebra

The way a system behaves under elimination tells you the geometry directly. If you reach a line such as $0=0$, one of the equations was redundant, and the solution has a free parameter, so the planes share a line or a whole plane. If you reach a false statement such as $0=5$, the equations are inconsistent, so the planes have no common point. If you reach three independent equations for three unknowns, the planes meet at a single point. It is a good idea to write a short sentence in your answer that links the algebra to the geometry, because the question often asks for this explicitly, and an answer that merely states the number of solutions may not earn the final mark.

Using technology wisely

Graphical calculators and computer algebra systems can find inverses and solve systems in an instant, and in the examination you may use a calculator to check or to carry out a long calculation. However, a question that says "show that" or "find, by algebra" expects to see the working. Use your calculator to check the answer that you have found by hand, and to find the inverse of a three by three matrix when the question allows it, but write down the matrix you entered and the matrix that you obtained. Examiners cannot award marks for a number that appears from nowhere, and a mistake in typing an entry is easy to make.

Looking ahead

The ideas of this chapter lead directly to eigenvalues and eigenvectors, where you look for the special directions that a matrix leaves unchanged. They also lead to the study of vector spaces and linear algebra at university, to numerical methods for solving huge systems with thousands of equations, and to the matrix methods used in statistics and machine learning. A thorough grasp of inverses, determinants and the meaning of a singular matrix will be a great help in all of those areas.

Did you know?

The idea of a determinant appeared in China more than two thousand years ago, in a book called The Nine Chapters on the Mathematical Art, which solved systems of linear equations by a method very similar to the one we now call Gaussian elimination. In Europe, Gottfried Leibniz and Seki Takakazu of Japan found determinants independently in the 1680s. The word matrix, from the Latin for womb, was introduced by James Joseph Sylvester in 1850, and his friend Arthur Cayley developed the algebra of matrices, including the inverse, in a famous paper of 1858.

Otto's tip

When $\det = 0$, look for a combination of equations equal to another. This shows at once whether the system is consistent, and finish with a sentence describing the planes, because marks are given for the interpretation.