Worksheet - Inverse matrices and systems of equations

Year: Further Maths · 20

Show your working.

Question 1

Question 1 – The inverse of a 2 by 2 matrix *(Section A · Skills practice)*

For $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the inverse is $\mathbf{A}^{-1} = \dfrac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ provided $ad - bc \neq 0$.

a. Find the inverse of $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$ and check by multiplying.

b. Find the inverse of $\begin{pmatrix} 4 & -2 \\ 3 & 1 \end{pmatrix}$.

c. Show that $\begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix}$ has no inverse.

d. Find the values of k for which $\begin{pmatrix} k & 3 \\ 4 & k+1 \end{pmatrix}$ is singular.

Question 2

Question 2 – Solving two equations with an inverse *(Section A · Skills practice)*

Write each pair of simultaneous equations as a matrix equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ and solve using $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

a. $3x + y = 7$ and $5x + 2y = 12$

b. $4x - 2y = 10$ and $3x + y = 5$

c. Explain what goes wrong if you try this method for $2x + 6y = 4$ and $x + 3y = 5$.

Question 3

Question 3 – Matrix equations *(Section A · Skills practice)*

The matrices are $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}$.

a. Find $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$.

b. Solve $\mathbf{A}\mathbf{X} = \mathbf{C}$ for $\mathbf{X}$.

c. Find $(\mathbf{A}\mathbf{B})^{-1}$ in two ways: directly, and as $\mathbf{B}^{-1}\mathbf{A}^{-1}$.

Question 4

Question 4 – The inverse of a 3 by 3 matrix *(Section A · Skills practice)*

Let .

a. Find $\det \mathbf{M}$ by expanding along the first row.
b. Find the matrix of cofactors, and hence the adjugate $\operatorname{adj} \mathbf{M}$ (the transpose of the cofactor matrix).
c. Write down $\mathbf{M}^{-1}$.
d. Check that $\mathbf{M}\mathbf{M}^{-1} = \mathbf{I}$ by multiplying the first row of $\mathbf{M}$ by each column of $\mathbf{M}^{-1}$.

Question 5

Question 5 – Solving three equations *(Section A · Skills practice)*

Solve the system $x + y + z = 6$, $2x - y + z = 3$, $x + 2y - z = 2$ using an inverse matrix.

a. Write the system as $\mathbf{A}\mathbf{x} = \mathbf{b}$ and find $\det \mathbf{A}$.
b. Find the matrix of cofactors and the adjugate of $\mathbf{A}$.
c. Write down $\mathbf{A}^{-1}$.
d. Hence solve the system and check your answer in all three equations.

Question 6

Question 6 – Systems with no inverse *(Section A · Skills practice)*

Consider the system $x + y + z = 1$, $x + 2y + 3z = 2$, $2x + 3y + 4z = k$.

a. Show that the determinant of the coefficient matrix is zero.
b. Explain why the system is inconsistent unless $k = 3$.
c. When $k = 3$, find the general solution and describe it geometrically.
d. State what happens when $k = 5$.

Question 7

Question 7 – Prices at the café *(Section B · Real-life practice)*

A café sells coffee, tea and cake. Over three days the owner records the number of each item sold and the takings.

Day	Coffees	Teas	Cakes	Takings (£)
Monday	10	8	6	70
Tuesday	12	5	8	78
Wednesday	7	10	4	57

a. Let a coffee cost £c, a tea £t and a cake £k. Write the information as a matrix equation.
b. Show that the determinant of the coefficient matrix is -26, so the prices can be found uniquely.
c. Use the inverse matrix to find c, t and k.
d. Check your answer using the Monday takings.

Question 8

Question 8 – Planning a menu *(Section B · Real-life practice)*

A dietitian mixes three ingredients. Per portion of 100 g, ingredient X gives 20 g of protein, 5 g of carbohydrate and 2 g of fat; ingredient Y gives 10 g, 30 g and 4 g; ingredient Z gives 5 g, 10 g and 8 g. A meal must contain exactly 100 g of protein, 115 g of carbohydrate and 46 g of fat.

a. Let the meal contain x, y, z portions of X, Y, Z. Write down the three equations.
b. Show that the coefficient matrix has determinant 3600.
c. Solve for x, y, z.
d. Find the total mass of the meal in grams.

Question 9

Question 9 – Currents in a circuit *(Section B · Real-life practice)*

In an electric circuit with three branches, the currents I_1, I_2, I_3 (in amperes) satisfy Kirchhoff's laws: $I_1 - I_2 - I_3 = 0$, $4I_1 + 6I_2 = 18$ and $6I_2 + 3I_3 = 12$.

a. Write the equations as a matrix equation.
b. Show that the coefficient matrix has determinant 6.
c. Find the three currents.
d. Verify that the current law holds for your answers.

Question 10

Question 10 – Traffic through a town *(Section B · Real-life practice)*

Three roads meet at a one-way system. The flows of traffic (cars per minute) are x, y and z and they satisfy $x + y = 50$, $y + z = 70$, $x - z = -20$.

a. Write the system as a matrix equation and show that the determinant of the coefficient matrix is zero.
b. Show that the system is consistent and find its general solution in terms of y.
c. Flows cannot be negative. Find the range of possible values of y.
d. The council measures $y = 30$. Find x and z.

Question 11

Question 11 – Where the roof planes meet *(Section B · Real-life practice)*

Three sloping roof surfaces of a modern house lie in the planes $x + 2y + z = 7$, $2x - y + z = 2$ and $x + y - z = 1$, where lengths are in metres.

a. Write the equations in matrix form and find the determinant of the coefficient matrix.
b. Explain what a non-zero determinant says about the three planes.
c. Find the coordinates of the point where all three planes meet.
d. Verify that this point satisfies all three equations.

Question 12

Question 12 – Undoing an image transformation *(Section B · Real-life practice)*

A photo-editing program transforms the pixel at position (x, y) to (x', y') using the matrix $\mathbf{M} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

a. Find $\mathbf{M}^{-1}$ and use it to find the original position of the pixel now at $(5, 3)$.
b. Find the area scale factor of the transformation. What does it say about the area of the picture?
c. A second effect uses $\mathbf{N} = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$. A triangle of area 3 cm^2 is transformed by $\mathbf{N}$. Find the area of its image.
d. Verify with the triangle with vertices $(0, 0)$, $(2, 0)$ and $(0, 3)$.

Question 13

Question 13 – A secret code *(Section B · Real-life practice)*

A simple code replaces pairs of letters by numbers ($\mathrm{A} = 0, \mathrm{B} = 1, \ldots, \mathrm{Z} = 25$), multiplies by the matrix $\mathbf{K} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$, and takes the remainders after division by 26.

a. Encode the pair HI.
b. Show that $\mathbf{K}^{-1} = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$, which gives $\begin{pmatrix} 1 & 24 \\ 25 & 3 \end{pmatrix}$ when the entries are written modulo 26.
c. Decode the pair PB.
d. Why is it essential that $\det \mathbf{K} = 1$ (or at least has no common factor with 26)?

Question 14

Question 14 – Two industries *(Section B · Real-life practice)*

An economy has two industries. To produce one unit of output, industry 1 needs 0.2 units from industry 1 and 0.4 units from industry 2; industry 2 needs 0.3 units from industry 1 and 0.1 units from industry 2. Customers demand 100 units from industry 1 and 200 units from industry 2. The outputs x_1, x_2 must satisfy $(\mathbf{I}-\mathbf{A})\mathbf{x}=\mathbf{d}$ where $\mathbf{A}=\begin{pmatrix}0.2 & 0.3\\ 0.4 & 0.1\end{pmatrix}$.

a. Write down $\mathbf{I}-\mathbf{A}$ and show that its determinant is 0.6.
b. Find $(\mathbf{I}-\mathbf{A})^{-1}$.
c. Find the outputs x_1 and x_2 needed.
d. Explain why the total output is larger than the demand.

Question 15

Question 15 – Three planes and a parameter *(Section C · Problem solving)*

Consider the system $x+y+z=1$, $x+2y+3z=2$, $2x+3y+kz=3$, where k is a constant.

a. Show that the determinant of the coefficient matrix is $k-4$.
b. State what this tells you about the solutions when $k\neq 4$.
c. Show that when $k=4$ the system is consistent and find the general solution.
d. Describe the geometry of the three planes when $k=4$.

Question 16

Question 16 – A prism of planes *(Section C · Problem solving)*

The planes Π_1: $x+y+z=1$, Π_2: $2x+y-z=2$ and Π_3: $3x+2y=5$ are given.

a. Show that no two of the planes are parallel.
b. Show that the coefficient matrix is singular.
c. Show that the system has no solution.
d. Describe the arrangement of the three planes and sketch it.

Question 17

Question 17 – An inverse with a parameter *(Section C · Problem solving)*

The matrix depends on a constant k.

a. Show that $\det\mathbf{A}=5-2k$ and state the value of k for which $\mathbf{A}$ is singular.
b. For $k=1$, find $\mathbf{A}^{-1}$.
c. Hence solve when $k=1$.

d. Show that when $k = \frac{5}{2}$ the system has no solution.

Question 18

Question 18 – Using a matrix identity *(Section C · Problem solving)*

The matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$ is given.

a. Show that $\mathbf{A}^2 = 2\mathbf{A} + \mathbf{I}$.
b. Hence show that $\mathbf{A}(\mathbf{A} - 2\mathbf{I}) = \mathbf{I}$ and deduce that $\mathbf{A}^{-1} = \mathbf{A} - 2\mathbf{I}$.
c. Check this against the formula for the inverse of a 2×2 matrix.
d. Show that $\mathbf{A}^3 = 5\mathbf{A} + 2\mathbf{I}$ and calculate $\mathbf{A}^3$.

Question 19

Question 19 – A transformation and its inverse *(Section C · Problem solving)*

A linear transformation T maps the point $(1, 0)$ to $(2, 1)$ and the point $(0, 1)$ to $(-1, 3)$.

a. Write down the matrix $\mathbf{M}$ of T.
b. Find $\det \mathbf{M}$ and the area of the image of a triangle of area 4.
c. Find $\mathbf{M}^{-1}$.
d. Find the point that is mapped by T to $(3, 5)$, and verify your answer.

Question 20

Question 20 – Challenge: the determinant of the adjugate *(Section D · Challenge)*

For a 3×3 matrix $\mathbf{A}$ the adjugate satisfies $\mathbf{A}\,\mathrm{adj}\,\mathbf{A} = (\det \mathbf{A})\mathbf{I}$.

a. Use $\det(\mathbf{PQ}) = \det \mathbf{P} \det \mathbf{Q}$ to show that $\det(\mathrm{adj}\,\mathbf{A}) = (\det \mathbf{A})^2$ when $\det \mathbf{A} \neq 0$.
b. Verify the result for the matrix $\mathbf{M}$ of Question 4, where $\det \mathbf{M} = 13$ and

.

c. Show that $\mathrm{adj}(\mathrm{adj}\,\mathbf{A}) = (\det \mathbf{A})\mathbf{A}$ for a 3×3 matrix with non-zero determinant.
d. Explain why the result in part (a) does not hold in the same form for a 2×2 matrix.