Answers - Inverse matrices and systems of equations

Year: Further Maths

Answer to Question 1

a. Determinant $= 6 - 5 = 1$. Inverse $= \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$. Check:
$\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 6-5 & -3+3 \\ 10-10 & -5+6 \end{pmatrix} = \mathbf{I}$ ✓

b. Determinant $= 4 + 6 = 10$. Inverse $= \frac{1}{10}\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}$.

c. Determinant $= 6 - 6 = 0$, so the matrix is **singular** and has no inverse (its columns are parallel).

d. Singular when $k(k+1) - 12 = 0$, i.e. $k^2 + k - 12 = (k+4)(k-3) = 0$, so $k = 3$ **or** $k = -4$.

Answer to Question 2

a. $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 12 \end{pmatrix}$. $\mathbf{x} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 7 \\ 12 \end{pmatrix} = \begin{pmatrix} 14-12 \\ -35+36 \end{pmatrix}$. $x = 2$, $y = 1$.
Check: $6 + 1 = 7$ ✓, $10 + 2 = 12$ ✓.

b. $\mathbf{x} = \frac{1}{10}\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 10 \\ 5 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 20 \\ -10 \end{pmatrix}$. $x = 2$, $y = -1$. Check: $8 + 2 = 10$ ✓, $6 - 1 = 5$ ✓.

c. The matrix $\begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix}$ has determinant 0, so there is no inverse. The equations are inconsistent: the first says $x + 3y = 2$ and the second says $x + 3y = 5$, so there is **no solution**.

Answer to Question 3

a. $\det \mathbf{A} = 5 - 6 = -1$, so $\mathbf{A}^{-1} = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}$. $\det \mathbf{B} = 6$, so $\mathbf{B}^{-1} = \frac{1}{6}\begin{pmatrix} 3 & 0 \\ -1 & 2 \end{pmatrix}$.

b. Multiply on the left: $\mathbf{X} = \mathbf{A}^{-1}\mathbf{C} = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} -1 & -22 \\ 1 & 13 \end{pmatrix}$.

c. $\mathbf{AB} = \begin{pmatrix} 4 & 6 \\ 11 & 15 \end{pmatrix}$, determinant $60 - 66 = -6$. Directly:
$\frac{1}{-6}\begin{pmatrix} 15 & -6 \\ -11 & 4 \end{pmatrix} = \begin{pmatrix} -\frac{5}{2} & 1 \\ \frac{11}{6} & -\frac{2}{3} \end{pmatrix}$. And

$\mathbf{B}^{-1}\mathbf{A}^{-1} = \frac{1}{6}\begin{pmatrix} 3 & 0 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} -15 & 6 \\ 11 & -4 \end{pmatrix}$, the same. **Reverse order of the inverses.**

Answer to Question 4

a. $\det \mathbf{M} = 1(1 \cdot 1 - 3 \cdot 0) - 2(0 \cdot 1 - 3 \cdot 2) + 0 = 1 + 12 = \mathbf{13}$
b. Cofactors: $C_{11} = 1$, $C_{12} = -(0 - 6) = 6$, $C_{13} = 0 - 2 = -2$; $C_{21} = -(2 - 0) = -2$, $C_{22} = 1 - 0 = 1$, $C_{23} = -(0 - 4) = 4$; $C_{31} = 6 - 0 = 6$, $C_{32} = -(3 - 0) = -3$, $C_{33} = 1$. Cofactor matrix [redacted], so .

c.

d. First row of $\mathbf{M}$ is $(1, 2, 0)$. Times column 1: $\frac{1}{13}(1 + 12 + 0) = 1$; column 2: $\frac{1}{13}(-2 + 2 + 0) = 0$; column 3: $\frac{1}{13}(6 - 6 + 0) = 0$. The first row of $\mathbf{I}$. ✓

Answer to Question 5

a. , .

$\det \mathbf{A} = 1(1 - 2) - 1(-2 - 1) + 1(4 + 1) = -1 + 3 + 5 = \mathbf{7}$
b. Cofactors: $C_{11} = -1$, $C_{12} = 3$, $C_{13} = 5$, $C_{21} = 3$, $C_{22} = -2$, $C_{23} = -1$, $C_{31} = 2$, $C_{32} = 1$, $C_{33} = -3$. The adjugate is the transpose: [redacted].

c.

d. . $x = 1$, $y = 2$, $z = 3$. Check: $1 + 2 + 3 = 6$ ✓; $2 - 2 + 3 = 3$ ✓; $1 + 4 - 3 = 2$ ✓.

Answer to Question 6

a. $\det = 1(8 - 9) - 1(4 - 6) + 1(3 - 4) = -1 + 2 - 1 = \mathbf{0}$
b. Adding the first two equations gives $2x + 3y + 4z = 3$. So the third equation says the same thing as the sum of the first two when $k = 3$; for any other k the

equations contradict each other, and there is **no solution**.

c. Subtract the first two equations: $y + 2z = 1$, so $y = 1 - 2z$. Then $x = 1 - y - z = z$. The solution is $(x, y, z) = (t, 1 - 2t, t)$ for any t: the three planes meet along a **line** (a sheaf).

d. For $k = 5$ the third equation conflicts with the sum of the first two ($3 \neq 5$): **no solution**. The three planes form a triangular prism.

Answer to Question 7

a. 

b. $\det = 10(20 - 80) - 8(48 - 56) + 6(120 - 35) = -600 + 64 + 510 = -26$, which is not zero.

c. Using the inverse (calculator or cofactors) : **coffee £3, tea £2, cake £4**.

d. Monday: $10 \times 3 + 8 \times 2 + 6 \times 4 = 30 + 16 + 24 = 70$ ✓. Tuesday: $36 + 10 + 32 = 78$ ✓. Wednesday: $21 + 20 + 16 = 57$ ✓.

Answer to Question 8

a. Protein: $20x + 10y + 5z = 100$. Carbohydrate: $5x + 30y + 10z = 115$. Fat: $2x + 4y + 8z = 46$.

b. $\det = 20(240 - 40) - 10(40 - 20) + 5(20 - 60) = 4000 - 200 - 200 =$ **3600**

c. : **3 portions of X, 2 of Y, 4 of Z**. Check protein: $60 + 20 + 20 = 100$ ✓.

d. $3 + 2 + 4 = 9$ portions of 100 g: **900 g**.

Answer to Question 9

a. 

b. $\det = 1(18 - 0) - (-1)(12 - 0) + (-1)(24 - 0) = 18 + 12 - 24 =$ **6**

c. Inverse gives : $I_1 = 3$ **A,** $I_2 = 1$ **A,** $I_3 = 2$ **A**. Check: $12 + 6 = 18$ ✓;

$6 + 6 = 12$ ✓.

d. $I_1 - I_2 - I_3 = 3 - 1 - 2 = 0$ ✓, so the current entering the junction equals the current leaving.

Answer to Question 10

a. [redacted] has determinant $1(-1 - 0) - 1(0 - 1) = -1 + 1 = \mathbf{0}$.

b. Subtracting the first equation from the second gives $z - x = 20$, i.e. $x - z = -20$, which is the third equation. So the system is **consistent** with infinitely many solutions: $x = 50 - y$, $z = 70 - y$, with y free.

c. $x \geq 0$ gives $y \leq 50$; $z \geq 0$ gives $y \leq 70$; $y \geq 0$. So $0 \leq y \leq 50$.

d. $x = 50 - 30 = \mathbf{20}$ and $z = 70 - 30 = \mathbf{40}$ cars per minute.

Answer to Question 11

a. [redacted].

$\det = 1(1 - 1) - 2(-2 - 1) + 1(2 + 1) = 0 + 6 + 3 = \mathbf{9}$

b. Since $\det \neq 0$ the system has a unique solution: the three planes meet at **exactly one point** (like the corner of a room).

c. $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = (1, 2, 2)$. The planes meet at $(1, 2, 2)$.

d. $1 + 4 + 2 = 7$ ✓; $2 - 2 + 2 = 2$ ✓; $1 + 2 - 2 = 1$ ✓.

Answer to Question 12

a. $\det \mathbf{M} = 2 - 1 = 1$, so $\mathbf{M}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$. $\mathbf{M}^{-1} \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$: the original pixel was at $(2, 1)$. Check: $\mathbf{M} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$ ✓.

b. The area scale factor is $|\det \mathbf{M}| = \mathbf{1}$: areas are preserved (the picture is sheared and stretched but keeps its area).

c. $|\det \mathbf{N}| = 6$, so the image has area $6 \times 3 = \mathbf{18\ cm^2}$.

d. Vertices map to $(0, 0)$, $(6, 0)$ and $(3, 6)$. Area $= \frac{1}{2}|6 \times 6 - 0 \times 3| = \mathbf{18}$ ✓.

Answer to Question 13

a. H = 7 and I = 8. $\mathbf{K} \begin{pmatrix} 7 \\ 8 \end{pmatrix} = \begin{pmatrix} 21 + 16 \\ 7 + 8 \end{pmatrix} = \begin{pmatrix} 37 \\ 15 \end{pmatrix} \equiv \begin{pmatrix} 11 \\ 15 \end{pmatrix}$, which is the pair **LP**.

b. $\det \mathbf{K} = 3 - 2 = 1$, so $\mathbf{K}^{-1} = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$. Adding 26 to negative entries: $-2 \equiv 24$ and $-1 \equiv 25$.

c. P = 15 and B = 1. $\begin{pmatrix} 1 & 24 \\ 25 & 3 \end{pmatrix} \begin{pmatrix} 15 \\ 1 \end{pmatrix} = \begin{pmatrix} 39 \\ 378 \end{pmatrix} \equiv \begin{pmatrix} 13 \\ 14 \end{pmatrix}$, which is the pair **NO**.

d. To decode, the matrix must have an inverse that uses only whole numbers modulo 26. If the determinant shared a factor with 26, there would be no such inverse, and different messages would give the same code.

Answer to Question 14

a. $\mathbf{I} - \mathbf{A} = \begin{pmatrix} 0.8 & -0.3 \\ -0.4 & 0.9 \end{pmatrix}$, $\det = 0.72 - 0.12 = \mathbf{0.6}$

b. $(\mathbf{I} - \mathbf{A})^{-1} = \dfrac{1}{0.6} \begin{pmatrix} 0.9 & 0.3 \\ 0.4 & 0.8 \end{pmatrix}$

c. $\mathbf{x} = \dfrac{1}{0.6} \begin{pmatrix} 90 + 60 \\ 40 + 160 \end{pmatrix} = \dfrac{1}{0.6} \begin{pmatrix} 150 \\ 200 \end{pmatrix}$, so $x_1 = 250$ **and** $x_2 \approx 333.3$ units.

d. Part of each industry's output is used up as input by the industries themselves, so more than the final demand has to be produced.

Answer to Question 15

a. $\det = 1(2k - 9) - 1(k - 6) + 1(3 - 4) = 2k - 9 - k + 6 - 1 = k - 4$

b. When $k \neq 4$ the determinant is non-zero, so there is a **unique solution**: the planes meet at a single point.

c. For $k = 4$, the third equation $2x + 3y + 4z = 3$ is the sum of the first two equations ($1 + 2 = 3$), so it adds no new information: consistent. Subtracting the first two gives $y + 2z = 1$. Putting $z = t$: $y = 1 - 2t$ and $x = 1 - y - z = t$. $(x, y, z) = (t, 1 - 2t, t)$.

d. The three planes are all different and none are parallel, and they meet along a common **line** with direction vector $(1, -2, 1)$ (a sheaf of planes).

Answer to Question 16

a. The normals are $(1, 1, 1)$, $(2, 1, -1)$ and $(3, 2, 0)$. No normal is a multiple of another, so **no two planes are parallel**.

b. $\det = 1(0 + 2) - 1(0 + 3) + 1(4 - 3) = 2 - 3 + 1 = \mathbf{0}$

c. Adding Π_1 and Π_2 gives $3x + 2y = 3$, but Π_3 says $3x + 2y = 5$: a contradiction, so **no solution**.

d. Each pair of planes meets in a line, and the three lines are parallel (all with direction $(2, -3, 1)$ here) but never coincide: the planes form a **triangular prism**.

Answer to Question 17

a. $\det = 1(1-0) - 2(0-2) + k(0-2) = 1 + 4 - 2k = 5 - 2k$. Singular when $k = \frac{5}{2}$.

b. For $k = 1$, $\det = 3$. The adjugate is [redacted], so

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c. , so $(x, y, z) = (0, -1, 3)$. Check: $0 - 2 + 3 = 1$ ✓;

$-1 + 3 = 2$ ✓; $0 + 0 + 3 = 3$ ✓.

d. For $k = \frac{5}{2}$ the equations are $x + 2y + \frac{5}{2}z = 1$, $y + z = 2$, $2x + z = 3$. From the second $y = 2 - z$; the first becomes $x + 4 - 2z + \frac{5}{2}z = 1$, i.e. $x + \frac{1}{2}z = -3$, so $2x + z = -6$. This contradicts the third equation $2x + z = 3$. **No solution.**

Answer to Question 18

a. $\mathbf{A}^2 = \begin{pmatrix} 1+2 & 1+1 \\ 2+2 & 2+1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$ and $2\mathbf{A} + \mathbf{I} = \begin{pmatrix} 2+1 & 2 \\ 4 & 2+1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$ ✓

b. $\mathbf{A}^2 - 2\mathbf{A} = \mathbf{I}$, i.e. $\mathbf{A}(\mathbf{A} - 2\mathbf{I}) = \mathbf{I}$. Hence $\mathbf{A}^{-1} = \mathbf{A} - 2\mathbf{I} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$.

c. $\det \mathbf{A} = 1 - 2 = -1$, so $\mathbf{A}^{-1} = \frac{1}{-1}\begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$ ✓

d. $\mathbf{A}^3 = \mathbf{A} \cdot \mathbf{A}^2 = \mathbf{A}(2\mathbf{A} + \mathbf{I}) = 2\mathbf{A}^2 + \mathbf{A} = 2(2\mathbf{A} + \mathbf{I}) + \mathbf{A} = 5\mathbf{A} + 2\mathbf{I} = \begin{pmatrix} 7 & 5 \\ 10 & 7 \end{pmatrix}$.

Answer to Question 19

a. The columns are the images of the unit vectors: $\mathbf{M} = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$.

b. $\det \mathbf{M} = 6 + 1 = \mathbf{7}$. The image has area $7 \times 4 = \mathbf{28}$.

c. $\mathbf{M}^{-1} = \frac{1}{7}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$

d. $\mathbf{M}^{-1}\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 14 \\ 7 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, the point $(2, 1)$. Check: $\mathbf{M}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4-1 \\ 2+3 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ ✓.

Answer to Question 20

a. Take determinants of both sides of $\mathbf{A}\operatorname{adj}\mathbf{A} = (\det\mathbf{A})\mathbf{I}$:
$\det\mathbf{A}\cdot\det(\operatorname{adj}\mathbf{A}) = (\det\mathbf{A})^3\det\mathbf{I} = (\det\mathbf{A})^3$. Divide by $\det\mathbf{A}$:
$\det(\operatorname{adj}\mathbf{A}) = (\det\mathbf{A})^2$.

b. $\det(\operatorname{adj}\mathbf{M}) = 1(1+12)+2(6-6)+6(24+2) = 13+0+156 = 169 = 13^2$ ✓.

c. Apply the identity to the matrix $\operatorname{adj}\mathbf{A}$:
$\operatorname{adj}\mathbf{A}\cdot\operatorname{adj}(\operatorname{adj}\mathbf{A}) = \det(\operatorname{adj}\mathbf{A})\mathbf{I} = (\det\mathbf{A})^2\mathbf{I}$. Multiply on the left by $\mathbf{A}$:
$(\det\mathbf{A})\mathbf{I}\cdot\operatorname{adj}(\operatorname{adj}\mathbf{A}) = (\det\mathbf{A})^2\mathbf{A}$, so $\operatorname{adj}(\operatorname{adj}\mathbf{A}) = (\det\mathbf{A})\mathbf{A}$.

d. For 2×2 the identity gives $\det\mathbf{A}\det(\operatorname{adj}\mathbf{A}) = (\det\mathbf{A})^2$, so $\det(\operatorname{adj}\mathbf{A}) = \det\mathbf{A}$ to the power 1, not 2. In general the power is $n-1$ for an $n\times n$ matrix.