Test answers - Hyperbolic functions

Year: Further Maths · Duration: 60 minutes · /20

Answer to Question 1 (4 marks)

Method: replace each hyperbolic function by its exponential definition, $\sinh x = \dfrac{e^x - e^{-x}}{2}$ and $\cosh x = \dfrac{e^x + e^{-x}}{2}$, and use $e^{\ln a} = a$. The identity $\cosh^2 x - \sinh^2 x = 1$ is the hyperbolic version of $\cos^2 x + \sin^2 x = 1$. Mark scheme as shown in brackets.

a. Square $e^x + e^{-x}$: $e^{2x} + 2 + e^{-2x}$, then divide by 4 and multiply by 2.
$2\cosh^2 x - 1 = \dfrac{(e^x + e^{-x})^2}{2} - 1$ [1] $= \dfrac{e^{2x} + e^{-2x}}{2} = \cosh 2x$ [1]

b. Use part a) to obtain a quadratic in $c = \cosh x$; remember that $\cosh x \geq 1$ for every real x. $2c^2 - 1 = 3c - 2$ gives $2c^2 - 3c + 1 = 0$, $(2c - 1)(c - 1) = 0$ [1]. $\cosh x \geq 1$ so $\cosh x = 1$ and $x = 0$ [1] The root $c = \frac{1}{2}$ is rejected.

Answer to Question 2 (4 marks)

Method: replace each hyperbolic function by its exponential definition, $\sinh x = \dfrac{e^x - e^{-x}}{2}$ and $\cosh x = \dfrac{e^x + e^{-x}}{2}$, and use $e^{\ln a} = a$. The identity $\cosh^2 x - \sinh^2 x = 1$ is the hyperbolic version of $\cos^2 x + \sin^2 x = 1$. Multiply by e^x to obtain a quadratic in e^x. Mark scheme as shown in brackets.

a. $\dfrac{2(e^x - e^{-x}) + (e^x + e^{-x})}{2} = 2$ [1] gives $3e^x - e^{-x} = 4$ [1]. With $u = e^x$:
$3u^2 - 4u - 1 = 0$ [1]. $u = \dfrac{4 + \sqrt{28}}{6} = \dfrac{2 + \sqrt{7}}{3}$, so $x = \ln \dfrac{2 + \sqrt{7}}{3}$ (negative root rejected) [1]

Check: with $e^x = \frac{2+\sqrt{7}}{3} \approx 1.549$, $2\sinh x + \cosh x \approx 0.903 + 1.097 = 2$.

Answer to Question 3 (4 marks)

Method: $\dfrac{d}{dx}\sinh x = \cosh x$ and $\dfrac{d}{dx}\cosh x = \sinh x$ (no minus sign, unlike the trigonometric case), so $\int \cosh ax\, dx = \dfrac{1}{a}\sinh ax + c$ and $\int \sinh ax\, dx = \dfrac{1}{a}\cosh ax + c$. The chain, product and quotient rules work as usual. Mark scheme as shown in brackets.

a. Take $u = x$ and $v = \sinh 2x$, so $v' = 2\cosh 2x$. $\dfrac{dy}{dx} = \sinh 2x + 2x\cosh 2x$ [2: 1 product rule, 1 chain rule]

b. Differentiate u^2 with $u = \cosh 3x$, then use $2\sinh A\cosh A = \sinh 2A$ with $A = 3x$. $\dfrac{dy}{dx} = 2\cosh 3x \cdot 3\sinh 3x = 6\sinh 3x\cosh 3x$ [1] $= 3\sinh 6x$ [1] So $k = 3$.

Answer to Question 4 (5 marks)

Method: $\dfrac{d}{dx}\sinh x = \cosh x$ and $\dfrac{d}{dx}\cosh x = \sinh x$ (no minus sign, unlike the trigonometric case), so $\int \cosh ax\,dx = \dfrac{1}{a}\sinh ax + c$ and $\int \sinh ax\,dx = \dfrac{1}{a}\cosh ax + c$. The chain, product and quotient rules work as usual. Mark scheme as shown in brackets.

a. $\left[\operatorname{arsinh}\dfrac{x}{2}\right]_0^2$ [1] $= \operatorname{arsinh} 1 = \ln(1+\sqrt{2})$ [1]

b. Let $u = x$, $dv = \sinh x\,dx$: $[x\cosh x]_0^1 - \int_0^1 \cosh x\,dx$ [1] $= \cosh 1 - [\sinh x]_0^1$ [1] $= \cosh 1 - \sinh 1 = e^{-1}$ [1]

Answers: a) $\ln(1+\sqrt{2}) \approx 0.881$; b) the integral equals $e^{-1} \approx 0.368$, since $\cosh 1 - \sinh 1 = e^{-1}$ by the definitions.

Answer to Question 5 (3 marks)

Method: in a model, identify what each variable represents, substitute the given values, and use the properties of $\cosh$ (minimum 1 at 0, even function) and $\tanh$ (between -1 and 1, tends to 1) to interpret the answers. Mark scheme as shown in brackets.

a. Substitute $x = 15$; by symmetry both posts have the same height. $10\cosh 1.5 =$ **23.5 m** [1] The lowest point of the cable is 10 m high, at $x = 0$.

b. $\cosh$ is even, so integrate from 0 to 15 and double. $\displaystyle\int_{-15}^{15} \cosh\frac{x}{10}dx = 2\left[10\sinh\frac{x}{10}\right]_0^{15}$ [1] $= 20\sinh 1.5 =$ **42.6 m** [1] So the cable is about 42.6 m long, compared with a 30 m gap between the posts.