Lesson - Hyperbolic functions

Year: Further Maths

Hyperbolic functions are built from e^x and e^{-x}. They behave like trigonometric functions, with similar identities and calculus, but with some changes of sign.

Starter activity

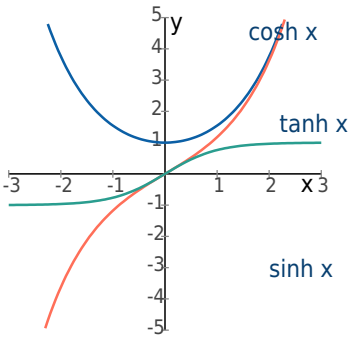
Hold a necklace or a length of chain by its two ends and let it hang. The curve it makes looks like a parabola, but it is not. Galileo thought it was a parabola, but a more careful analysis in 1691 by Leibniz, Huygens and Johann Bernoulli showed that it is the graph of $y = a\cosh\dfrac{x}{a}$. Sketch the graphs of e^x and e^{-x} on the same axes and add them together. Divide by two. What shape do you get? What is its lowest point, and is the graph symmetrical?

Definitions and graphs

Definition: hyperbolic functions

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}, \quad \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{2x} - 1}{e^{2x} + 1}.$$

Also $\operatorname{sech} x = \frac{1}{\cosh x}$, $\operatorname{cosech} x = \frac{1}{\sinh x}$ and $\coth x = \frac{1}{\tanh x}$.



Graphs of y = cosh x (blue), y = sinh x (coral) and y = tanh x (green, horizontal asymptotes y = ±1).

Key fact: shape and symmetry

$\cosh x \geq 1$ and is even. $\sinh x$ and $\tanh x$ are odd, and -1. The graph of $\cosh x$ is the shape of a hanging chain, the catenary $y = a\cosh\frac{x}{a}$.

Identities

Rule: basic identities

$\cosh^2 x - \sinh^2 x = 1$. Dividing gives $1 - \tanh^2 x = \operatorname{sech}^2 x$ and $\coth^2 x - 1 = \operatorname{cosech}^2 x$. Double angle: $\sinh 2x = 2\sinh x \cosh x$ and $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$. Addition: $\sinh(A \pm B) = \sinh A \cosh B \pm \cosh A \sinh B$ and $\cosh(A \pm B) = \cosh A \cosh B \pm \sinh A \sinh B$.

Note: Osborn's rule

To convert a trigonometric identity to a hyperbolic one, replace each $\cos$ by $\cosh$ and each $\sin$ by $\sinh$, but change the sign wherever there is a product (or implied product) of two sines. For example $1 + \tan^2 x = \sec^2 x$ becomes $1 - \tanh^2 x = \operatorname{sech}^2 x$.

Example 1

Show that $\cosh x + \sinh x = e^x$ and deduce $\cosh^2 x - \sinh^2 x = 1$.

Adding the definitions gives e^x. Also $\cosh x - \sinh x = e^{-x}$. Multiplying: $(\cosh x + \sinh x)(\cosh x - \sinh x) = e^x e^{-x} = 1$, which is $\cosh^2 x - \sinh^2 x = 1$.

Solving equations

Method: hyperbolic equations

1. Write each function in terms of e^x and put $u = e^x$.
2. Multiply through by u to get a quadratic in u.
3. Solve it and reject any $u \leq 0$, since $e^x > 0$.
4. Take logarithms to find x, leaving exact answers where possible.

Alternatively use identities to get a quadratic in $\sinh x$ or $\cosh x$.

Example 2

Solve $\cosh x = 2$.

$e^x + e^{-x} = 4 \Rightarrow u^2 - 4u + 1 = 0 \Rightarrow u = 2 \pm \sqrt{3}$. Both are positive, so $x = \pm\ln(2 + \sqrt{3})$ (since $(2 + \sqrt{3})(2 - \sqrt{3}) = 1$).

Watch out

$\cosh x = -1$ has no solution, and only a positive value of e^x is valid. Never take $\ln$ of a negative number.

Inverse hyperbolic functions

Rule: logarithmic forms

$\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2+1}\right)$, $\operatorname{arcosh} x = \ln\left(x + \sqrt{x^2-1}\right)$ $(x \geq 1)$, $\operatorname{artanh} x = \frac{1}{2}\ln\frac{1+x}{1-x}$ $(|x|$

Note: proof for arsinh

Let $y = \operatorname{arsinh} x$, so $x = \sinh y = \frac{e^y - e^{-y}}{2}$. Then $e^{2y} - 2xe^y - 1 = 0$, so $e^y = x + \sqrt{x^2+1}$ (the positive root).

Example 3

Solve $\sinh x = 2$**.**

$x = \operatorname{arsinh} 2 = \ln\left(2 + \sqrt{5}\right) \approx 1.444$. Check:

$\sinh x = \frac{1}{2}\left(2 + \sqrt{5} - \frac{1}{2+\sqrt{5}}\right) = \frac{1}{2}\left(2 + \sqrt{5} - (\sqrt{5} - 2)\right) = 2.$

Differentiation and integration

Key fact: derivatives

$f(x)$	$f'(x)$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$ (no minus sign)
$\tanh x$	$\operatorname{sech}^2 x$
$\operatorname{arsinh} x$	$\frac{1}{\sqrt{x^2+1}}$
$\operatorname{arcosh} x$	$\frac{1}{\sqrt{x^2-1}}$
$\operatorname{artanh} x$	$\frac{1}{1-x^2}$

Rule: standard integrals

$\int \sinh x\,dx = \cosh x$, $\int \cosh x\,dx = \sinh x$, $\int \frac{dx}{\sqrt{x^2+a^2}} = \operatorname{arsinh}\frac{x}{a}$, $\int \frac{dx}{\sqrt{x^2-a^2}} = \operatorname{arcosh}\frac{x}{a}$ for $x > a$. Complete the square for quadratics inside roots. For $\sinh^2 x$ and $\cosh^2 x$ use $\cosh 2x = 1 + 2\sinh^2 x$ or $2\cosh^2 x - 1$.

Example 4

Evaluate $\displaystyle\int_0^3 \frac{dx}{\sqrt{x^2+9}}$.

$\left[\operatorname{arsinh}\frac{x}{3}\right]_0^3 = \operatorname{arsinh} 1 = \ln(1+\sqrt{2})$.

Example 5

Find $\displaystyle\int \sinh x \cosh x \, dx$.

With $u = \sinh x$, $du = \cosh x \, dx$, the integral is $\frac{1}{2}\sinh^2 x + c$. The form $\frac{1}{4}\cosh 2x + c'$ is also correct, because $\sinh^2 x = \frac{1}{2}(\cosh 2x - 1)$.

Watch out

- Writing $\frac{d}{dx}\cosh x = -\sinh x$; that is the trigonometric sign pattern.
- Forgetting the minus sign in $\cosh^2 x - \sinh^2 x = 1$.
- Using $\operatorname{arcosh} x$ for x; it is not defined.
- Mixing up arsinh (plus sign under the root) and arcosh (minus sign).

Note: even and odd functions

The function $\cosh x$ is even, which means that $\cosh(-x) = \cosh x$, so its graph is symmetric about the y-axis. The function $\sinh x$ is odd, which means that $\sinh(-x) = -\sinh x$, so its graph is symmetric under a half turn about the origin. Together they split the exponential function into its even part and its odd part, since $e^x = \cosh x + \sinh x$ and $e^{-x} = \cosh x - \sinh x$. This is the reason for the identity $\cosh^2 x - \sinh^2 x = 1$: multiplying the two equations gives $e^x e^{-x} = 1$. Recognising symmetry also helps when sketching graphs and when evaluating integrals over a symmetric interval, since the integral of an odd function from $-a$ to a is zero.

Key fact: graphs and ranges

The graph of $y = \cosh x$ has a minimum value of 1 at $x = 0$, is always positive, and grows rapidly for large $|x|$ like $\frac{1}{2}e^{|x|}$. The graph of $y = \sinh x$ passes through the origin with gradient 1 and increases for all x, so it has an inverse function defined for all real numbers. The graph of $y = \tanh x$ also passes through the origin with gradient 1, but it levels off at the horizontal asymptotes $y = 1$ and $y = -1$, so its range is the open interval from -1 to 1. Because $\cosh x$ is not one-to-one, its inverse $\operatorname{arcosh} x$ is defined only for $x \geq 1$ and gives the positive value.

Worked example 5: proving an identity with Osborn's rule

Use Osborn's rule to write down an identity for $\cosh 3x$ starting from $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, and check it when $x = \ln 2$.

Replacing every function by its hyperbolic counterpart gives $\cosh 3x = 4\cosh^3 x - 3\cosh x$ with no sign changes, since there is no product of two sines. To verify, take $x = \ln 2$: $\cosh \ln 2 = \frac{5}{4}$, so the right-hand side is $4 \cdot \frac{125}{64} - \frac{15}{4} = \frac{125}{16} - \frac{60}{16} = \frac{65}{16}$. The left side is $\cosh(3\ln 2) = \cosh \ln 8 = \frac{8+1/8}{2} = \frac{65}{16}$. The two agree, so the identity is correct.

Method: solving equations with hyperbolic functions

1. If there are several hyperbolic functions, use identities to express everything in terms of one function, for example $\cosh^2 x = 1 + \sinh^2 x$.
2. If the equation then becomes a quadratic, solve it, and reject any value that lies outside the range of the function: $\cosh x \geq 1$, and $|\tanh x| < 1$.
3. Alternatively, replace each function by its exponential definition and let $u = e^x$, remembering that $u > 0$. This leads to a quadratic in u.
4. Convert the solution back using $x = \ln u$ or by an inverse hyperbolic function, and give exact logarithmic answers if requested.

Worked example 6: an equation which gives two roots

Solve $\sinh^2 x - 2\cosh x + 2 = 0$.

Replace $\sinh^2 x$ by $\cosh^2 x - 1$. The equation becomes $\cosh^2 x - 2\cosh x + 1 = 0$, which factorises as $(\cosh x - 1)^2 = 0$. Hence $\cosh x = 1$, and the only solution is $x = 0$. This is a repeated root, which tells you that the graph of the left-hand side touches the axis at $x = 0$ without crossing it. A common error is to square-root incorrectly or to miss that $\cosh x = 1$ has just one solution, whereas $\cos\theta = 1$ would have infinitely many.

Watch out: signs in the hyperbolic identities

- The identity is $\cosh^2 x - \sinh^2 x = 1$, with a minus sign. Do not confuse it with $\cos^2\theta + \sin^2\theta = 1$.
- The derivative of $\cosh x$ is $+\sinh x$ with no minus sign, unlike the derivative of $\cos x$.
- The derivative of $\tanh x$ is $\operatorname{sech}^2 x$, which is $1 - \tanh^2 x$.
- Osborn's rule changes a sign only when a term contains a product (or implied product) of two sines, such as $\sin^2\theta$ or $\tan^2\theta$.
- Do not forget that $\operatorname{arcosh} x$ needs $x \geq 1$, and that $\operatorname{artanh} x$ needs $|x| < 1$.

Worked example 7: integrating with a hyperbolic substitution

Show that $\int_0^3 \sqrt{x^2+9}\,dx$ can be found using $x = 3\sinh u$, and evaluate it to 3 significant figures.

With $x = 3\sinh u$ we have $dx = 3\cosh u\,du$ and $\sqrt{x^2+9} = 3\cosh u$. The integrand becomes $9\cosh^2 u = \frac{9}{2}(1+\cosh 2u)$. The upper limit $x = 3$ corresponds to $\sinh u = 1$, so $u = \operatorname{arsinh} 1 \approx 0.8814$. The integral is $\frac{9}{2}\left[u + \frac{1}{2}\sinh 2u\right]_0^{0.8814}$. Since $\sinh 2u = 2\sinh u\cosh u = 2\sqrt{2}$, this equals $\frac{9}{2}\left(0.8814 + \sqrt{2}\right) = \frac{9}{2}(2.2956) = 10.33$. Checking with the formula $\frac{x}{2}\sqrt{x^2+9} + \frac{9}{2}\operatorname{arsinh}\frac{x}{3}$ gives $\frac{3}{2}\sqrt{18} + \frac{9}{2}(0.8814) = 6.364 + 3.966 = 10.33$.

Applications of hyperbolic functions

Hyperbolic functions occur naturally whenever a quantity grows or decays exponentially in two directions. The shape of a hanging cable, the Gateway Arch in St Louis, and many suspension bridge cables follow the catenary curve, because a flexible chain under its own weight takes the shape of the graph of the hyperbolic cosine. The graph of $\tanh$ describes saturating processes, such as the speed of a falling object subject to air resistance, the magnetisation of a material, and the activation function used in artificial neural networks. In physics, the rapidity of special relativity is the inverse hyperbolic tangent of the speed divided by the speed of light.

Method: revision plan for this chapter

1. Learn the definitions of $\sinh$, $\cosh$ and $\tanh$ in terms of e^x, and practise deriving their graphs.
2. Learn the identities and practise Osborn's rule on at least five trigonometric identities.
3. Practise solving equations both by identities and by the substitution $u = e^x$.
4. Derive the logarithmic forms of the inverse functions once yourself so that you remember how they arise.
5. Do mixed integration and differentiation questions, including the substitutions $x = a\sinh u$ and $x = a\cosh u$.

Why the hyperbola and not the circle

The names of the hyperbolic functions reflect a close analogy with the trigonometric functions. For the circular functions, the point with coordinates cosine and sine of an angle lies on the unit circle, and the angle measures twice the area of the sector cut off. For the hyperbolic functions, the point with coordinates hyperbolic cosine and hyperbolic sine of a number lies on the right-hand branch of the unit hyperbola, and the number measures twice the area of the corresponding hyperbolic sector. Because the trigonometric functions are built from imaginary exponentials and the hyperbolic functions from real ones, the two families are linked by the relations $\cosh ix = \cos x$

and $\sinh ix = i \sin x$. This is the deeper reason why Osborn's rule works: converting a trigonometric identity into a hyperbolic one is equivalent to replacing x by ix, and the sign changes arise from the powers of i.

Worked example 8: simplifying an expression

Show that $\dfrac{\cosh x - \sinh x}{\cosh x + \sinh x} = e^{-2x}$.

We use the facts that $\cosh x + \sinh x = e^{x}$ and $\cosh x - \sinh x = e^{-x}$, which follow directly from the definitions by adding and subtracting the two exponentials. Then the fraction is $\dfrac{e^{-x}}{e^{x}} = e^{-2x}$. This type of simplification is useful because it avoids expanding or squaring, and it shows clearly why the function $\tanh x$ can be written as $\dfrac{e^{2x} - 1}{e^{2x} + 1}$. It also explains why the graph of $\tanh$ approaches 1 so quickly: the term e^{-2x} becomes negligible after only a few units.

Reading graphs and asymptotes

When sketching graphs involving hyperbolic functions, mark the key features carefully. For $y = \cosh x$, label the minimum point $(0, 1)$ and note that the curve is symmetric. For $y = \sinh x$, mark the origin and show that the gradient there is 1, and that there is no maximum or minimum. For $y = \tanh x$, draw the horizontal asymptotes $y = 1$ and $y = -1$ as dashed lines, and show that the curve is steepest at the origin. For the inverse functions, remember that their graphs are reflections of the originals in the line $y = x$: the graph of $\operatorname{arsinh} x$ grows slowly like a logarithm, the graph of $\operatorname{arcosh} x$ starts at the point $(1, 0)$ with a vertical tangent, and the graph of $\operatorname{artanh} x$ has vertical asymptotes at $x = 1$ and $x = -1$.

Worked example 9: a derivative that uses the product rule and chain rule together

Find the stationary points of $y = x - \tanh x$ for $x \geq 0$ and decide whether the function is increasing.

The derivative is $\dfrac{dy}{dx} = 1 - \operatorname{sech}^2 x$. Using the identity $1 - \operatorname{sech}^2 x = \tanh^2 x$, this equals $\tanh^2 x$, which is zero only at $x = 0$ and is positive for all other values of x. Therefore the function is increasing for all x, with a stationary point of inflexion at the origin where the gradient is momentarily zero but does not change sign. The graph of $y = x - \tanh x$ therefore starts at the origin, rises slowly at first, and then behaves like the straight line $y = x - 1$ for large x, since $\tanh x$ tends to 1.

Watch out: exact and decimal answers

Questions on hyperbolic functions often ask for answers in exact logarithmic form,

such as $\ln(2+\sqrt{3})$, and the examiner will not accept a decimal approximation unless the question asks for one. When you convert from an inverse hyperbolic function to a logarithm, simplify the expression inside the logarithm as far as possible, but do not worry about writing it in a unique form; any equivalent expression receives full marks. Remember that $\ln\dfrac{1}{a} = -\ln a$, which is useful for turning negative solutions into the same form as positive ones. When both positive and negative solutions are possible, as in $\cosh x = 2$, make sure that you give both.

Linking hyperbolic functions to series and complex numbers

The exponential function has a power series that holds for every real number, and the hyperbolic functions inherit series from it. Taking the even terms of the series for e^x gives $\cosh x = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \cdots$, and taking the odd terms gives $\sinh x = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \cdots$. These resemble the series for cosine and sine, but with all the signs positive. You will meet these series in the chapter on Maclaurin expansions. For small values of x the first term is a very good approximation, so $\sinh x \approx x$ and $\cosh x \approx 1 + \frac{1}{2}x^2$, which explains why the graph of the catenary looks like a parabola near its lowest point.

Exam technique for hyperbolic questions

Questions on this topic often begin with a "show that" step, for example proving an identity or deriving a logarithmic form. Always set out the proof from the definitions in terms of exponentials, because this is the method that the mark scheme expects, and write the final line exactly as the result stated in the question. When solving an equation, state the range of the function in order to justify rejecting a root, since this earns a mark. When integrating, check the form of the integral against the formula booklet: if the square root has a plus sign inside, the answer involves arsinh, and if it has a minus sign, the answer involves arcosh. Finally, always check an answer by estimating it numerically with a calculator, using the exponential definitions if necessary.

Worked example 10: a physical interpretation

A cable has the shape $y = a\cosh\dfrac{x}{a}$. Show that the gradient at any point equals the length of cable measured from the lowest point to that point divided by the constant a.

The gradient is $\dfrac{dy}{dx} = \sinh\dfrac{x}{a}$. The length of cable from the lowest point at $x = 0$ to the point with coordinate x is the integral of the square root of one plus the gradient squared, and since this square root equals the hyperbolic cosine, the length is

$s = \int_0^x \cosh \frac{t}{a}\, dt = a \sinh \frac{x}{a}$. Dividing by a gives $\frac{s}{a} = \sinh \frac{x}{a}$, which is the gradient. This remarkable property explains why a hanging cable is so useful in engineering: the slope at the support is directly proportional to the length of cable hanging below that point, which is a measure of the weight the support must carry. The tension in the cable at any point is also proportional to the height of that point above the base line.

Final advice

The hyperbolic functions reward careful, tidy work. Because the identities are so close to the familiar trigonometric ones, it is easy to write a wrong sign without noticing, so make a habit of checking each identity with a simple value such as $x = 0$ or $x = \ln 2$. Practise the derivations of the logarithmic forms, since a question may ask for them as a proof. Use the definitions in terms of exponentials whenever you are unsure, because they always work. With a secure understanding of this chapter you will find that integrals involving square roots of quadratics, which looked difficult before, become routine applications of a single substitution.

Did you know?

The word hyperbolic comes from the hyperbola, because the point $(\cosh t, \sinh t)$ lies on the hyperbola $x^2 - y^2 = 1$ in the same way as $(\cos t, \sin t)$ lies on the circle $x^2 + y^2 = 1$. The hyperbolic functions were introduced by the Italian Jesuit Vincenzo Riccati in the 1760s and developed by Johann Heinrich Lambert. The first accurate analysis of the hanging chain was one of the great problems of the late seventeenth century, solved independently by Leibniz, Huygens and Johann Bernoulli in 1691, only a few years after the invention of calculus.

Otto's tip

When stuck, rewrite everything in terms of e^x. It always works.