# Worksheet - Hyperbolic functions

Year: Further Maths · 20

Show your working.

## Question 1

**Question 1 – Exact values** *(Section A · Skills practice)*

Recall that $\sinh x = \dfrac{e^x - e^{-x}}{2}$, $\cosh x = \dfrac{e^x + e^{-x}}{2}$ and $\tanh x = \dfrac{\sinh x}{\cosh x}$. Evaluate each of the following exactly, without a calculator.

a. $\sinh(\ln 3)$
b. $\cosh(\ln 3)$
c. $\tanh(\ln 3)$
d. Verify that $\cosh^2(\ln 3) - \sinh^2(\ln 3) = 1$.

## Question 2

**Question 2 – Hyperbolic identities** *(Section A · Skills practice)*

Use the definitions in terms of exponentials, or Osborn's rule, for each part.

a. Prove that $\sinh 2x = 2\sinh x \cosh x$.
b. Prove that $\cosh 2x = 2\cosh^2 x - 1$.
c. Use Osborn's rule to convert $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ into an identity for $\cosh 3x$.
d. Given that $\cosh x = \dfrac{5}{3}$ with $x > 0$, find $\sinh x$ and $\tanh x$.

## Question 3

**Question 3 – Solving equations** *(Section A · Skills practice)*

Solve each equation, giving exact answers in terms of natural logarithms.

a. $\cosh x = 2$
b. $\sinh x = -\dfrac{3}{4}$
c. $\cosh 2x - 3\cosh x + 2 = 0$
d. $4\cosh x + 5\sinh x = 3$

## Question 4

**Question 4 – Inverse hyperbolic functions** *(Section A · Skills practice)*

The inverse hyperbolic functions have the logarithmic forms

$\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2+1}\right)$, $\operatorname{arcosh} x = \ln\left(x + \sqrt{x^2-1}\right)$ for $x \geq 1$, and $\operatorname{artanh} x = \frac{1}{2}\ln\dfrac{1+x}{1-x}$ for $|x|$.

a. Find $\operatorname{arsinh} 2$ and $\operatorname{arcosh} 3$ in exact form, and to 3 decimal places.
b. Find $\operatorname{artanh} \frac{1}{2}$ in exact form.
c. Prove the formula for $\operatorname{arsinh} x$ by letting $y = \operatorname{arsinh} x$ and solving $x = \sinh y$ for $e^y$.

## Question 5

**Question 5 – Differentiation** *(Section A · Skills practice)*

Differentiate each function with respect to $x$. Use $\dfrac{d}{dx}\sinh x = \cosh x$, $\dfrac{d}{dx}\cosh x = \sinh x$ and $\dfrac{d}{dx}\tanh x = \operatorname{sech}^2 x$.

a. $\sinh 3x$
b. $x^2 \cosh x$
c. $\ln(\cosh x)$
d. $\tanh 2x$ and $\sinh^2 x$

## Question 6

**Question 6 – Integration** *(Section A · Skills practice)*

Find each integral. Use $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}} = \operatorname{arsinh}\frac{x}{a}$ and $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = \operatorname{arcosh}\frac{x}{a}$.

a. $\displaystyle\int \cosh 2x \, dx$
b. $\displaystyle\int \sinh^2 x \, dx$ (use $\cosh 2x = 1 + 2\sinh^2 x$)
c. $\displaystyle\int_0^1 \tanh x \, dx$
d. $\displaystyle\int_2^4 \frac{dx}{\sqrt{x^2-4}}$

## Question 7

**Question 7 – The power line** *(Section B · Real-life practice)*

A power line hangs between two pylons 60 m apart and takes the shape of the catenary $y = 20\cosh\dfrac{x}{20}$, where $x$ and $y$ are in metres and the origin is vertically below the lowest point of the cable at ground level directly under its midpoint (so $-30 \leq x \leq 30$).

a. Find the lowest height of the cable above the ground.
b. Find the height of the cable at each pylon, to 2 decimal places.
c. Find the length of the cable, using the fact that the arc length of a curve is $\int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$.
d. Find the angle that the cable makes with the horizontal at the pylon.

## Question 8

**Question 8 – A hyperbolic arch** *(Section B · Real-life practice)*

A garden arch is modelled by $y = 40 - 5\left(\cosh\frac{x}{10} - 1\right)$ for $y \geq 0$, with lengths in metres.

a. Find the height of the arch at its centre.
b. Show that the arch meets the ground where $\cosh\frac{x}{10} = 9$.
c. Hence find the width of the arch at ground level, in exact form and to 1 decimal place.
d. Find the height of the arch at $x = 15$.

## Question 9

**Question 9 – Falling with air resistance** *(Section B · Real-life practice)*

A sphere falling from rest through air with resistance proportional to the square of the speed has speed $v = 40\tanh\frac{9.8t}{40}$ m s$^{-1}$ after $t$ seconds.

a. State the terminal velocity and explain how you can tell from the formula.
b. Find the speed after 3 seconds.
c. Show that the acceleration is $\frac{dv}{dt} = 9.8\,\mathrm{sech}^2\frac{9.8t}{40}$ and find its value at $t = 0$.
d. After what time is the speed 99% of the terminal velocity?

## Question 10

**Question 10 – Waves on the sea** *(Section B · Real-life practice)*

The speed $v$ of a water wave of wavelength $\lambda$ metres in water of depth $d$ metres satisfies $v^2 = \frac{g\lambda}{2\pi}\tanh\frac{2\pi d}{\lambda}$, with $g = 9.8$ m s$^{-2}$.

a. Find the speed of a wave of wavelength 20 m in water 5 m deep.
b. Find the speed of the same wave in very deep water, using $\tanh \to 1$.
c. By what percentage is the wave slowed by the shallow water?
d. Explain why waves slow down as they approach a beach.

## Question 11

**Question 11 – Cooling fins on a computer** *(Section B · Real-life practice)*

A metal fin of length 0.5 m cools a computer processor. The temperature excess above the room along the fin is $T(x) = T_0 \dfrac{\cosh m(L - x)}{\cosh mL}$, where $x$ is the distance from the processor, $L = 0.5$ m, $m = 2$ m$^{-1}$ and $T_0 = 40^\circ$C is the excess at the base.

a. Verify that $T(0) = T_0$.
b. Find the temperature excess at the tip $x = L$.
c. Find the excess halfway along the fin, to 1 decimal place.
d. Show that $T$ is decreasing along the fin, using the derivative.

## Question 12

**Question 12 – Adding velocities** *(Section B · Real-life practice)*

In special relativity, velocities in a straight line combine by adding rapidities. A velocity $v$ is written $v = c \tanh \phi$, where $c$ is the speed of light and $\phi$ is the rapidity. Rapidities add: $\phi = \phi_1 + \phi_2$.

a. Find $\phi_1$ and $\phi_2$ for $v = 0.6c$ and $v = 0.8c$, using $\operatorname{artanh}$.
b. Find $\phi_1 + \phi_2$ and the corresponding speed as a fraction of $c$.
c. Prove that $\tanh(a + b) = \dfrac{\tanh a + \tanh b}{1 + \tanh a \tanh b}$ and use it to check part (b).
d. Explain why the combined speed is less than $c$ although $0.6 + 0.8 > 1$.

## Question 13

**Question 13 – How far has it fallen?** *(Section B · Real-life practice)*

For the sphere in Question 9, the distance fallen is the integral of the speed.

a. Show that $\displaystyle\int 40 \tanh \frac{9.8t}{40}\, dt = \frac{1600}{9.8} \ln \cosh \frac{9.8t}{40}$.
b. Find the distance fallen in the first 10 seconds, to the nearest metre.
c. Compare with the distance $40 \times 10 = 400$ m that would be covered at terminal speed throughout, and explain the difference.

## Question 14

**Question 14 – Minimising a cost** *(Section B · Real-life practice)*

The daily cost, in thousands of pounds, of running a delivery depot is modelled by $C(x) = 3\cosh x - \sinh x$, where $x$ is a measure of the number of vans on the road.

a. Find $C'(x)$ and show that the stationary point satisfies $\tanh x = \frac{1}{3}$.
b. Find the exact value of $x$ at the stationary point.

c. Find the minimum cost exactly and justify that it is a minimum.
d. Give the minimum cost in pounds.

## Question 15

**Question 15 – The derivative of an inverse function** *(Section C · Problem solving)*

Let $y = \operatorname{arcosh} x$ for $x > 1$.

a. Show that $\cosh y = x$ and differentiate implicitly to show that $\dfrac{dy}{dx} = \dfrac{1}{\sinh y}$.
b. Use $\cosh^2 y - \sinh^2 y = 1$ to show that $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x^2 - 1}}$.
c. Hence write down $\displaystyle\int \frac{dx}{\sqrt{x^2 - 4}}$ and evaluate $\displaystyle\int_2^5 \frac{dx}{\sqrt{x^2 - 4}}$ to 4 decimal places.
d. Check the answer to part (c) using the logarithmic form of $\operatorname{arcosh}$.

## Question 16

**Question 16 – A substitution** *(Section C · Problem solving)*

Use the substitution $x = 2\sinh u$ to evaluate $\displaystyle\int_0^2 \sqrt{x^2 + 4}\, dx$.

a. Show that $dx = 2\cosh u\, du$ and $\sqrt{x^2 + 4} = 2\cosh u$.
b. Find the limits for $u$.
c. Show that the integral becomes $\displaystyle\int_0^{\operatorname{arsinh} 1} (2 + 2\cosh 2u) du$.
d. Hence find the exact value of the integral and its value to 3 decimal places.

## Question 17

**Question 17 – Combining sinh and cosh** *(Section C · Problem solving)*

Consider the equation $4\sinh x + 3\cosh x = 6$.

a. Write $4\sinh x + 3\cosh x$ in the form $R\sinh(x + \alpha)$ and find $R$ and $\tanh \alpha$.
b. Solve the equation exactly using $\alpha = \frac{1}{2}\ln 7$.
c. Check by solving with the substitution $u = e^x$.
d. Show that the equation $3\sinh x + 4\cosh x = 2$ has no real solution.

## Question 18

**Question 18 – Stationary points of a curve** *(Section C · Problem solving)*

The curve $C$ has equation $y = \sinh x - 2x$.

a. Find $\dfrac{dy}{dx}$ and show that the stationary points occur where $\cosh x = 2$.
b. Find the exact coordinates of the stationary points.
c. Use the second derivative to determine their nature.
d. Hence sketch the curve and state how many real roots the equation $\sinh x = 2x$ has.

## Question 19

**Question 19 – Area, length and volume of a catenary** *(Section C · Problem solving)*

The curve $y = \cosh x$ is drawn for $0 \le x \le 1$. All lengths are in the same units.

a. Find the exact area under the curve between $x = 0$ and $x = 1$.
b. Show that $1 + \left(\dfrac{dy}{dx}\right)^2 = \cosh^2 x$ and hence find the exact length of the curve.
c. What do you notice about your answers to (a) and (b)?
d. Find the volume formed when the area in (a) is rotated about the $x$-axis, in exact form and to 3 decimal places.

## Question 20

**Question 20 – Challenge: the half-angle substitution** *(Section D · Challenge)*

Let $t = \tanh\dfrac{x}{2}$.

a. Show that $\sinh x = \dfrac{2t}{1-t^2}$ and $\cosh x = \dfrac{1+t^2}{1-t^2}$.
b. Show that $\dfrac{dx}{dt} = \dfrac{2}{1-t^2}$.
c. Hence show that $\displaystyle\int \operatorname{sech} x\,dx = 2\arctan t + C$.
d. Evaluate $\displaystyle\int_0^{\infty} \operatorname{sech} x\,dx$ and check numerically that the area under $\operatorname{sech} x$ for $0 \le x \le 6$ is close to your answer.