Answers - Hyperbolic functions

Year: Further Maths

Answer to Question 1

Method: replace each hyperbolic function by its exponential definition, $\sinh x = \dfrac{e^x - e^{-x}}{2}$ and $\cosh x = \dfrac{e^x + e^{-x}}{2}$, and use $e^{\ln a} = a$. The identity $\cosh^2 x - \sinh^2 x = 1$ is the hyperbolic version of $\cos^2 x + \sin^2 x = 1$.

a. $e^{\ln 3} = 3$ and $e^{-\ln 3} = \frac{1}{3}$. $\sinh(\ln 3) = \dfrac{3 - \frac{1}{3}}{2} = \dfrac{8/3}{2} = \dfrac{4}{3}$ So $\sinh(\ln 3) = \frac{4}{3}$.

b. Use the same two values in the definition of cosh. $\cosh(\ln 3) = \dfrac{3 + \frac{1}{3}}{2} = \dfrac{10/3}{2} = \dfrac{5}{3}$ So $\cosh(\ln 3) = \frac{5}{3}$.

c. Divide the answers to a) and b). $\tanh(\ln 3) = \dfrac{4/3}{5/3} = \dfrac{4}{5}$ As expected, tanh lies between -1 and 1.

d. Substitute the answers to a) and b) into the identity. $\left(\dfrac{5}{3}\right)^2 - \left(\dfrac{4}{3}\right)^2 = \dfrac{25 - 16}{9} = \dfrac{9}{9} = \mathbf{1}$ ✓ This confirms the fundamental identity for this value of x.

Answer to Question 2

a. $2\sinh x \cosh x = 2 \cdot \dfrac{e^x - e^{-x}}{2} \cdot \dfrac{e^x + e^{-x}}{2} = \dfrac{e^{2x} - e^{-2x}}{2} = \sinh 2x$ ✓

b. $2\cosh^2 x - 1 = 2 \cdot \dfrac{e^{2x} + 2 + e^{-2x}}{4} - 1 = \dfrac{e^{2x} + e^{-2x}}{2} = \cosh 2x$ ✓

c. Osborn's rule: replace each trig function by the hyperbolic one and change the sign of any product of two sines. There are none in this identity, so $\cosh 3x = 4\cosh^3 x - 3\cosh x$.

d. $\sinh^2 x = \cosh^2 x - 1 = \dfrac{25}{9} - 1 = \dfrac{16}{9}$ and $x > 0$ so $\sinh x = \dfrac{4}{3}$; $\tanh x = \dfrac{4}{5}$.

Answer to Question 3

a. $e^x + e^{-x} = 4$, so $u^2 - 4u + 1 = 0$ with $u = e^x$, $u = 2 \pm \sqrt{3}$. $x = \pm\ln(2 + \sqrt{3})$ (since $(2 + \sqrt{3})(2 - \sqrt{3}) = 1$).

b. $e^x - e^{-x} = -\frac{3}{2}$, $2u^2 + 3u - 2 = 0$, $(2u-1)(u+2) = 0$. Only $u = \frac{1}{2}$ is positive, so $x = -\ln 2$.

c. $2\cosh^2 x - 1 - 3\cosh x + 2 = 0$, so $2\cosh^2 x - 3\cosh x + 1 = 0$, $(2\cosh x - 1)(\cosh x - 1) = 0$. Since $\cosh x \geq 1$, $\cosh x = 1$ and $x = 0$.

d. $\dfrac{4(u + 1/u) + 5(u - 1/u)}{2} = 3$ gives $9u - \dfrac{1}{u} = 6$, $9u^2 - 6u - 1 = 0$, $u = \dfrac{6 \pm \sqrt{72}}{18} = \dfrac{1 \pm \sqrt{2}}{3}$. Only the positive root is valid: $x = \ln \dfrac{1+\sqrt{2}}{3}$ (≈ -0.2).

Answer to Question 4

Method: to evaluate an inverse hyperbolic function exactly, substitute into its logarithmic form. To prove a logarithmic form, write $x = \sinh y$ in exponentials, multiply by e^y to get a quadratic in e^y, and reject the root that is not positive.

a. $\operatorname{arsinh} 2 = \ln(2 + \sqrt{5}) =$ **1.444**. $\operatorname{arcosh} 3 = \ln(3 + \sqrt{8}) = \ln(3 + 2\sqrt{2}) =$ **1.763**.

b. $\frac{1}{2}\ln \dfrac{3/2}{1/2} = \frac{1}{2}\ln 3$ (≈ 0.549).

c. $x = \dfrac{e^y - e^{-y}}{2}$ gives $e^{2y} - 2xe^y - 1 = 0$. With $w = e^y$: $w = x \pm \sqrt{x^2+1}$. Since $w > 0$ we take the plus sign, so $y = \ln\left(x + \sqrt{x^2+1}\right)$ ✓.

Answer to Question 5

Method: $\dfrac{d}{dx}\sinh x = \cosh x$ and $\dfrac{d}{dx}\cosh x = \sinh x$ (no minus sign, unlike the trigonometric case), so $\int \cosh ax\, dx = \dfrac{1}{a}\sinh ax + c$ and $\int \sinh ax\, dx = \dfrac{1}{a}\cosh ax + c$. The chain, product and quotient rules work as usual.

a. The inside function $3x$ gives a factor 3. Chain rule: $3\cosh 3x$

b. Take $u = x^2$ and $v = \cosh x$, so $u' = 2x$ and $v' = \sinh x$. Product rule: $2x\cosh x + x^2 \sinh x$ This can be written $x(2\cosh x + x\sinh x)$.

c. Use $\dfrac{d}{dx}\ln u = \dfrac{u'}{u}$ with $u = \cosh x$. $\dfrac{\sinh x}{\cosh x} = \tanh x$ So this is the reverse of $\int \tanh x\, dx = \ln\cosh x + c$.

d. Use the chain rule for both; the second uses the identity $\sinh 2x = 2\sinh x \cosh x$. $\dfrac{d}{dx}\tanh 2x = 2\operatorname{sech}^2 2x$. $\dfrac{d}{dx}\sinh^2 x = 2\sinh x\cosh x = \sinh 2x$.

Answer to Question 6

Method: $\dfrac{d}{dx}\sinh x = \cosh x$ and $\dfrac{d}{dx}\cosh x = \sinh x$ (no minus sign, unlike the

trigonometric case), so $\int \cosh ax\,dx = \frac{1}{a}\sinh ax + c$ and $\int \sinh ax\,dx = \frac{1}{a}\cosh ax + c$. The chain, product and quotient rules work as usual.

a. Integrate $\cosh$ to $\sinh$ and divide by 2. $\frac{1}{2}\sinh 2x + C$ Check: differentiating gives $\cosh 2x$.
b. Rearrange the double angle identity to make $\sinh^2 x$ the subject, then integrate term by term. $\sinh^2 x = \frac{1}{2}(\cosh 2x - 1)$, so the integral is $\frac{1}{4}\sinh 2x - \frac{x}{2} + C$
c. $\tanh x = \dfrac{\sinh x}{\cosh x}$ has the numerator equal to the derivative of the denominator. $[\ln \cosh x]_0^1 = \ln \cosh 1 =$ **0.434** So the exact value is $\ln(\cosh 1)$.
d. Use the standard result with $a = 2$, and $\operatorname{arcosh} 1 = 0$. $\left[\operatorname{arcosh} \frac{x}{2}\right]_2^4 = \operatorname{arcosh} 2 - \operatorname{arcosh} 1 = \ln(2 + \sqrt{3}) - 0 =$ **1.317** So the exact value is $\ln(2 + \sqrt{3})$.

Answer to Question 7

Method: in a model, identify what each variable represents, substitute the given values, and use the properties of $\cosh$ (minimum 1 at 0, even function) and $\tanh$ (between -1 and 1, tends to 1) to interpret the answers.

a. At $x = 0$, $y = 20\cosh 0 =$ **20 m**.
b. $y = 20\cosh\dfrac{30}{20} = 20\cosh 1.5 = 20 \times 2.3524 =$ **47.05 m** (so it rises 27.05 m above its lowest point).
c. Arc length $s = \int_{-30}^{30} \sqrt{1 + \sinh^2 \frac{x}{20}}\,dx = \int_{-30}^{30} \cosh\dfrac{x}{20}\,dx = 2\left[20\sinh\dfrac{x}{20}\right]_0^{30} = 40\sinh 1.5 =$ **85.17 m**.
d. $\tan\theta = \dfrac{dy}{dx} = \sinh\dfrac{30}{20} = 2.129$, so $\theta =$ **64.8°**

Answer to Question 8

Method: in a model, identify what each variable represents, substitute the given values, and use the properties of $\cosh$ (minimum 1 at 0, even function) and $\tanh$ (between -1 and 1, tends to 1) to interpret the answers.

a. At $x = 0$, $\cosh 0 = 1$, so $y =$ **40 m**.
b. $y = 0$ gives $5\left(\cosh\frac{x}{10} - 1\right) = 40$, so $\cosh\frac{x}{10} - 1 = 8$, $\cosh\frac{x}{10} = 9$.
c. $\frac{x}{10} = \pm\operatorname{arcosh} 9 = \pm\ln(9 + \sqrt{80})$. So the width is $2 \times 10\ln(9 + 4\sqrt{5}) = 20\ln(9 + 4\sqrt{5}) \approx$ **57.7 m**.
d. $y = 40 - 5(\cosh 1.5 - 1) = 40 - 5(1.3524) =$ **33.2 m**.

Answer to Question 9

Method: in a model, identify what each variable represents, substitute the given values, and use the properties of $\cosh$ (minimum 1 at 0, even function) and $\tanh$ (between -1 and 1, tends to 1) to interpret the answers.

a. As $t \to \infty$, $\tanh \to 1$, so $v \to$ **40 m s^{-1}**.
b. $v = 40 \tanh 0.735 = 40 \times 0.6263 =$ **25.1 m s^{-1}**.
c. $\dfrac{dv}{dt} = 40 \cdot \dfrac{9.8}{40} \operatorname{sech}^2 \dfrac{9.8t}{40} = 9.8 \operatorname{sech}^2 \dfrac{9.8t}{40}$. At $t = 0$ this is **9.8 m s^{-2}**, i.e. g.
d. $\tanh \dfrac{9.8t}{40} = 0.99$, so $\dfrac{9.8t}{40} = \operatorname{artanh} 0.99 = \frac{1}{2} \ln 199 = 2.647$, $t = \dfrac{2.647 \times 40}{9.8} =$ **10.8 s**.

Answer to Question 10

a. $\dfrac{2\pi \times 5}{20} = 1.5708$ and $\tanh 1.5708 = 0.9172$.
$v^2 = \dfrac{9.8 \times 20}{2\pi} \times 0.9172 = 31.19 \times 0.9172 = 28.61$, so $v =$ **5.35 m s^{-1}**.
b. $v^2 = 31.19$, $v =$ **5.59 m s^{-1}**.
c. $\dfrac{5.59 - 5.35}{5.59} =$ **4.2%**.
d. As d decreases, $\tanh \dfrac{2\pi d}{\lambda}$ decreases below 1, so v decreases: that is why waves bunch up and slow near the shore.

Answer to Question 11

Method: in a model, identify what each variable represents, substitute the given values, and use the properties of $\cosh$ (minimum 1 at 0, even function) and $\tanh$ (between -1 and 1, tends to 1) to interpret the answers.

a. Substitute $x = 0$. $T(0) = T_0 \dfrac{\cosh mL}{\cosh mL} = T_0 =$ **40°C**. So the formula gives the correct temperature at the base.
b. Substitute $x = L$, so $m(L - x) = 0$ and $mL = 1$. $T(L) = T_0 \dfrac{\cosh 0}{\cosh 1} = \dfrac{40}{1.543} =$ **25.9°C**. So the tip is about $25.9°$C above room temperature.
c. Substitute $x = 0.25$, so $m(L - x) = 0.5$. $x = 0.25$:
$T = 40 \dfrac{\cosh 0.5}{\cosh 1} = 40 \times \dfrac{1.1276}{1.5431} =$ **29.2°C**. So halfway along, the excess is about $29.2°$C.
d. Differentiate with the chain rule; the inside function $m(L - x)$ has derivative $-m$.
$T'(x) = -mT_0 \dfrac{\sinh m(L - x)}{\cosh mL}$ for $0 \le x$, so the temperature falls along the fin.

Since $\sinh$ is positive for positive arguments, the derivative is negative.

Answer to Question 12

a. $\phi_1 = \frac{1}{2}\ln\dfrac{1.6}{0.4} = \frac{1}{2}\ln 4 = \mathbf{0.693}$; $\phi_2 = \frac{1}{2}\ln\dfrac{1.8}{0.2} = \frac{1}{2}\ln 9 = \mathbf{1.099}$.

b. $\phi = 1.792$ and $\tanh 1.792 = \mathbf{0.9459}$, so $v = 0.946c$.

c. $\tanh(a+b) = \dfrac{\sinh a\cosh b + \cosh a\sinh b}{\cosh a\cosh b + \sinh a\sinh b}$; divide top and bottom by $\cosh a\cosh b$. Then $\dfrac{0.6+0.8}{1+0.48} = \dfrac{1.4}{1.48} = 0.9459$ ✓.

d. $\tanh$ is always less than 1, so rapidities of any size give speeds below c. Speeds do not add as ordinary numbers; rapidities do.

Answer to Question 13

a. Let $k = \dfrac{9.8}{40}$. $\int 40\tanh kt\,dt = \dfrac{40}{k}\ln\cosh kt = \dfrac{1600}{9.8}\ln\cosh kt$.

b. $s(10) = 163.27\ln\cosh 2.45$. $\cosh 2.45 = 5.8373$, $\ln 5.8373 = 1.7640$. So $s = \mathbf{288\ m}$ (to the nearest metre).

c. At terminal speed the sphere would have gone 400 m, which is more than 288 m, because it takes time to build up to that speed: it is slower during the acceleration phase.

Answer to Question 14

Method: $\dfrac{d}{dx}\sinh x = \cosh x$ and $\dfrac{d}{dx}\cosh x = \sinh x$ (no minus sign, unlike the trigonometric case), so $\int\cosh ax\,dx = \dfrac{1}{a}\sinh ax + c$ and $\int\sinh ax\,dx = \dfrac{1}{a}\cosh ax + c$. The chain, product and quotient rules work as usual.

a. $C'(x) = 3\sinh x - \cosh x = 0$ gives $\tanh x = \dfrac{1}{3}$.

b. $x = \operatorname{artanh}\frac{1}{3} = \frac{1}{2}\ln\dfrac{4/3}{2/3} = \frac{1}{2}\ln 2$.

c. At this point $e^x = \sqrt{2}$, so $\cosh x = \dfrac{\sqrt{2}+1/\sqrt{2}}{2} = \dfrac{3}{2\sqrt{2}}$ and $\sinh x = \dfrac{1}{2\sqrt{2}}$.
$C = \dfrac{9}{2\sqrt{2}} - \dfrac{1}{2\sqrt{2}} = \dfrac{8}{2\sqrt{2}} = 2\sqrt{2}$. $C'' = 3\cosh x - \sinh x = C > 0$, so it is a minimum.

d. $2\sqrt{2}\times 1000 \approx$ **£2828**.

Answer to Question 15

Method: $\frac{d}{dx}\sinh x = \cosh x$ and $\frac{d}{dx}\cosh x = \sinh x$ (no minus sign, unlike the trigonometric case), so $\int \cosh ax\,dx = \frac{1}{a}\sinh ax + c$ and $\int \sinh ax\,dx = \frac{1}{a}\cosh ax + c$. The chain, product and quotient rules work as usual.

a. By definition $\cosh y = x$. Differentiating, $\sinh y \frac{dy}{dx} = 1$, so $\frac{dy}{dx} = \frac{1}{\sinh y}$.

b. $\sinh y = \sqrt{\cosh^2 y - 1} = \sqrt{x^2 - 1}$ (positive since $y > 0$). So $\frac{dy}{dx} = \frac{1}{\sqrt{x^2-1}}$.

c. $\int \frac{dx}{\sqrt{x^2-4}} = \operatorname{arcosh}\frac{x}{2} + C$. From 2 to 5: $\operatorname{arcosh} 2.5 - \operatorname{arcosh} 1 = \operatorname{arcosh} 2.5 =$ **1.5668**.

d. $\ln\left(2.5 + \sqrt{6.25 - 1}\right) = \ln(2.5 + 2.2913) = \ln 4.7913 = 1.5668$ ✓.

Answer to Question 16

Method: for integrals containing $\sqrt{x^2 + a^2}$, substitute $x = a\sinh u$: the identity $1 + \sinh^2 u = \cosh^2 u$ removes the square root. Change the limits and use $\cosh^2 u = \frac{1}{2}(1 + \cosh 2u)$.

a. $\frac{dx}{du} = 2\cosh u$. $x^2 + 4 = 4\sinh^2 u + 4 = 4\cosh^2 u$, so $\sqrt{x^2+4} = 2\cosh u$.

b. $x = 0$ gives $u = 0$; $x = 2$ gives $\sinh u = 1$, $u = \operatorname{arsinh} 1 = \ln(1 + \sqrt{2})$.

c. The integrand becomes $2\cosh u \cdot 2\cosh u = 4\cosh^2 u = 2(1 + \cosh 2u) = 2 + 2\cosh 2u$.

d. $[2u + \sinh 2u]_0^{\operatorname{arsinh} 1} = 2\operatorname{arsinh} 1 + 2\sinh u\cosh u = 2\ln(1+\sqrt{2}) + 2\cdot 1\cdot\sqrt{2} = 2\sqrt{2} + 2\ln(1+\sqrt{2}) \approx 4.591$.

Answer: $\int_0^2 \sqrt{x^2+4}\,dx = 2\sqrt{2} + 2\ln(1+\sqrt{2}) \approx 4.591$.

Answer to Question 17

a. $R\sinh(x+\alpha) = R\sinh x\cosh\alpha + R\cosh x\sinh\alpha$. So $R\cosh\alpha = 4$, $R\sinh\alpha = 3$. Then $R^2 = 16 - 9 = 7$, $R = \sqrt{7}$, and $\tanh\alpha = \frac{3}{4}$.

b. $\alpha = \operatorname{artanh}\frac{3}{4} = \frac{1}{2}\ln\frac{7/4}{1/4} = \frac{1}{2}\ln 7$. Then $\sinh(x+\alpha) = \frac{6}{\sqrt{7}}$, $x + \alpha = \ln\left(\frac{6}{\sqrt{7}} + \sqrt{\frac{36}{7} + 1}\right) = \ln\frac{6+\sqrt{43}}{\sqrt{7}}$, so

$x = \ln\dfrac{6+\sqrt{43}}{\sqrt{7}} - \frac{1}{2}\ln 7 = \ln\dfrac{6+\sqrt{43}}{7} \approx 0.585.$

c. $\dfrac{4(u-1/u)+3(u+1/u)}{2} = 6$ gives $7u - \dfrac{1}{u} = 12$, $7u^2 - 12u - 1 = 0$,
$u = \dfrac{12+\sqrt{172}}{14} = \dfrac{6+\sqrt{43}}{7} = 1.794$. So $x = \ln 1.794 = 0.585$ ✓.

d. Here the coefficient of $\cosh x$ is larger than that of $\sinh x$, so write $4\cosh x + 3\sinh x = \sqrt{7}\cosh(x+\beta)$ with $\tanh\beta = \frac{3}{4}$. Since $\cosh(x+\beta) \geq 1$, the left side is at least $\sqrt{7} \approx 2.65$, which is greater than 2. So **there is no real solution**.

Answer to Question 18

a. $\dfrac{dy}{dx} = \cosh x - 2$, which is 0 when $\cosh x = 2$.

b. $x = \pm\ln(2+\sqrt{3})$. With $\cosh x = 2$, $\sinh x = \pm\sqrt{3}$. The points are $\left(\ln(2+\sqrt{3}),\ \sqrt{3} - 2\ln(2+\sqrt{3})\right) \approx (1.317, -0.902)$ and $\left(-\ln(2+\sqrt{3}),\ 2\ln(2+\sqrt{3}) - \sqrt{3}\right) \approx (-1.317, 0.902)$.

c. $\dfrac{d^2y}{dx^2} = \sinh x$. It is $+\sqrt{3} > 0$ at the first point (minimum) and $-\sqrt{3}$ at the second (maximum).

d. The curve passes through the origin, rises to a local maximum at $x = -1.317$, falls through a local minimum at $x = 1.317$, then rises. It crosses the x-axis at 0 and then twice more (at $x \approx \pm 2.18$), so $\sinh x = 2x$ has **3 real roots**.

Answer to Question 19

Method: $\dfrac{d}{dx}\sinh x = \cosh x$ and $\dfrac{d}{dx}\cosh x = \sinh x$ (no minus sign, unlike the trigonometric case), so $\int \cosh ax\,dx = \dfrac{1}{a}\sinh ax + c$ and $\int \sinh ax\,dx = \dfrac{1}{a}\cosh ax + c$. The chain, product and quotient rules work as usual.

a. $\displaystyle\int_0^1 \cosh x\,dx = \sinh 1 = \frac{e - e^{-1}}{2} \approx 1.175.$

b. $\dfrac{dy}{dx} = \sinh x$ and $1 + \sinh^2 x = \cosh^2 x$. Length $= \displaystyle\int_0^1 \cosh x\,dx = \sinh 1.$

c. The area and the arc length are **equal**, a special property of the catenary.

d. $V = \pi\displaystyle\int_0^1 \cosh^2 x\,dx = \frac{\pi}{2}\int_0^1 (1+\cosh 2x)dx = \frac{\pi}{2}\left[x + \frac{\sinh 2x}{2}\right]_0^1 =$

$$\frac{\pi}{2}\left(1+\frac{\sinh 2}{2}\right)\approx 4.418.$$

Answer to Question 20

a. $\sinh x = 2\sinh\frac{x}{2}\cosh\frac{x}{2} = \dfrac{2\sinh\frac{x}{2}\cosh\frac{x}{2}}{\cosh^2\frac{x}{2}-\sinh^2\frac{x}{2}}$ because the denominator equals 1.
Dividing top and bottom by $\cosh^2\frac{x}{2}$ gives $\dfrac{2t}{1-t^2}$. Similarly
$\cosh x = \dfrac{\cosh^2\frac{x}{2}+\sinh^2\frac{x}{2}}{\cosh^2\frac{x}{2}-\sinh^2\frac{x}{2}} = \dfrac{1+t^2}{1-t^2}$.

b. $t=\tanh\frac{x}{2}$ gives $\dfrac{dt}{dx} = \frac{1}{2}\operatorname{sech}^2\frac{x}{2} = \frac{1}{2}(1-t^2)$, so $\dfrac{dx}{dt} = \dfrac{2}{1-t^2}$.

c. $\operatorname{sech} x = \dfrac{1-t^2}{1+t^2}$, so
$$\int \operatorname{sech} x\,dx = \int \frac{1-t^2}{1+t^2}\cdot\frac{2}{1-t^2}dt = \int \frac{2}{1+t^2}dt = 2\arctan t + C.$$

d. As $x\to\infty$, $t\to 1$, so the integral from 0 is $2\arctan 1 - 2\arctan 0 = \dfrac{\pi}{2}\approx 1.5708$.
Numerically $\int_0^6 \operatorname{sech} x\,dx = 2\arctan(\tanh 3) = 1.5666$, very close to 1.5708.