Lesson - Further vectors

Year: Further Maths

Further vectors in three dimensions: the scalar and vector products, equations of lines and planes, and the angles, intersections and distances between them.

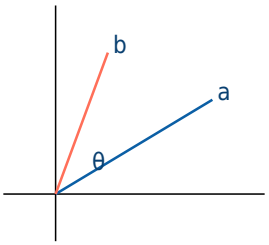
Starter activity

A drone is flying over a field. A radio mast stands at the origin of a coordinate system with the z-axis pointing straight up. At one moment the drone is at the point $(30, 40, 12)$ metres. How far is it from the mast? How far is it from the vertical line through the mast? If the drone flies in the direction $(3, 4, 0)$, is it moving towards the mast or away from it? Discuss how you could answer each question using a number calculated from two vectors, and decide whether you would need a product that gives a number or one that gives a vector.

Scalar and vector products

Definition: scalar (dot) product

$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = |\mathbf{a}||\mathbf{b}| \cos\theta$, where θ is the angle between the vectors.



Angle θ between two vectors a and b: a·b = |a||b|cos θ.

Key fact: using the dot product

Two vectors are perpendicular if and only if $\mathbf{a} \cdot \mathbf{b} = 0$. The angle between them satisfies $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$.

Definition: vector (cross) product

. It is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, with magnitude $|\mathbf{a}||\mathbf{b}| \sin\theta$

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Key fact: areas and volumes

$|\mathbf{a} \times \mathbf{b}|$ is the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$; a triangle has area $\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$. The scalar triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is the volume of the parallelepiped (a sixth of it for a tetrahedron).

Equation of a line

Rule: vector and Cartesian forms

Vector form: $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$, through $\mathbf{a}$ with direction $\mathbf{d}$. Cartesian form: $\frac{x - a_1}{d_1} = \frac{y - a_2}{d_2} = \frac{z - a_3}{d_3}$. A line through A and B has direction [illegible]. The angle between two lines is the angle between their direction vectors.

Method: do two lines meet?

1. Write each line with its own parameter, λ and μ.
2. Equate the x, y and z components.
3. Solve two of the equations for λ and μ.
4. Check the third equation. If it fails, the lines are **skew** (or parallel if the directions are multiples).

Rule: shortest distance between skew lines

For $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{d}_1$ and $\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{d}_2$:

$$d = \frac{|(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{d}_1 \times \mathbf{d}_2)|}{|\mathbf{d}_1 \times \mathbf{d}_2|}.$$

Equation of a plane

Rule: forms of a plane

- Normal form: $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$, Cartesian $n_1x + n_2y + n_3z = d$.
- Parametric: $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$ with two non-parallel directions; then $\mathbf{n} = \mathbf{b} \times \mathbf{c}$.
- Through three points A, B, C: [illegible].

Method: plane through three points

1. Find two edge vectors, [illegible] and [illegible].

2. Take their cross product to get the normal $\mathbf{n}$.
3. Write $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$ using any of the points.
4. Check that the other points satisfy the equation.

Angles, intersections and distances

Key fact: angles

- Between two planes: the angle between their normals; use $|\mathbf{n}_1 \cdot \mathbf{n}_2|$ for the acute angle.
- Between a line (direction $\mathbf{d}$) and a plane (normal $\mathbf{n}$): $\sin\varphi = \dfrac{|\mathbf{d} \cdot \mathbf{n}|}{|\mathbf{d}||\mathbf{n}|}$. It is sine, not cosine, because the normal is perpendicular to the plane.

Key fact: intersections

- Line meets plane: substitute $\mathbf{r} = \mathbf{a} + t\mathbf{d}$ into the plane equation and solve for t. No solution means parallel; $0 = 0$ means the line lies in the plane.
- Two planes meet in a line with direction $\mathbf{n}_1 \times \mathbf{n}_2$, plus one point found by setting a coordinate to 0.

Rule: distances

Point (x_1, y_1, z_1) to the plane $n_1x + n_2y + n_3z = d$: $\dfrac{|n_1x_1 + n_2y_1 + n_3z_1 - d|}{\sqrt{n_1^2 + n_2^2 + n_3^2}}$. Point P to a line through A with direction $\mathbf{d}$: . To reflect P in a plane, find the foot of the perpendicular, then go the same distance again.

Example 1

Find the distance from $P(1, 2, 3)$ to the plane $2x - 2y + z = 10$.

$\dfrac{|2 - 4 + 3 - 10|}{\sqrt{4 + 4 + 1}} = \dfrac{9}{3} = 3.$

Example 2

Find the angle between the planes $x + y = 1$ and $y + z = 1$.

The normals are $(1, 1, 0)$ and $(0, 1, 1)$, so $\cos\theta = \dfrac{1}{\sqrt{2}\sqrt{2}} = \dfrac{1}{2}$ and $\theta = 60°$.

Example 3

Find the foot of the perpendicular from the origin to $3x + 4y = 10$.

The distance is $\dfrac{10}{\sqrt{9+16}} = 2$. The foot is the point $t(3, 4, 0)$ with $3(3t) + 4(4t) = 25t = 10$, so it is $\left(\frac{6}{5}, \frac{8}{5}, 0\right)$.

Watch out

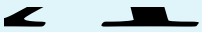
- Using $\cos$ instead of $\sin$ for the angle between a line and a plane.
- A sign error in the middle component of the cross product.
- Using the same parameter for both lines when finding an intersection.
- Giving an obtuse angle when the acute angle is asked for.

Key fact: properties of the two products

The scalar (dot) product $\mathbf{a} \cdot \mathbf{b}$ gives a number. It is commutative, distributes over addition, and satisfies $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$. It is zero when two non-zero vectors are perpendicular. The vector (cross) product $\mathbf{a} \times \mathbf{b}$ gives a vector perpendicular to both. It is anti-commutative, so $\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$, and it is zero when the two vectors are parallel. Its direction follows the right-hand rule, and its magnitude is the area of the parallelogram with the two vectors as sides. The two products combine in the scalar triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$, whose modulus is the volume of the parallelepiped with sides $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

Worked example 5: the shortest distance from a point to a line

Find the shortest distance from the point $P(3, 1, 2)$ to the line $\mathbf{r} = (1, 0, 1) + \lambda(2, 1, 2)$.

Let $A = (1, 0, 1)$ be the point on the line, so . The shortest distance is the perpendicular distance, and it equals . We compute , with magnitude $\sqrt{5}$. The direction vector has magnitude 3, so the distance is $\dfrac{\sqrt{5}}{3} \approx 0.745$. An alternative method is to write the foot of the perpendicular as a general point on the line and use the condition that the vector from P to this point is perpendicular to $\mathbf{d}$, which gives the same answer.

Method: finding the equation of a plane

1. If you know a point A and a normal $\mathbf{n}$, then $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$, and expanding gives the Cartesian equation $n_1x + n_2y + n_3z = d$.
2. If you know three points, form two vectors in the plane such as and , and obtain the normal as their cross product.
3. If you know a point and two directions, the normal is the cross product of the directions.
4. Always check by substituting every given point into the final equation.

Worked example 6: intersection of two planes

Find the line of intersection of the planes $x + y + z = 6$ and $x - y + 2z = 5$.

The direction of the line is perpendicular to both normals, so it is $\mathbf{d} = (1, 1, 1) \times (1, -1, 2) = (1 \cdot 2 - 1 \cdot (-1),\ 1 \cdot 1 - 1 \cdot 2,\ -1 - 1) = (3, -1, -2)$. To find a point on the line, choose a convenient value for one variable. Putting $z = 0$ gives $x + y = 6$ and $x - y = 5$, so $x = 5.5$ and $y = 0.5$. The line is therefore $\mathbf{r} = (5.5, 0.5, 0) + \lambda(3, -1, -2)$. Check a second point: with $\lambda = 1$ the point is $(8.5, -0.5, -2)$, and indeed $8.5 - 0.5 - 2 = 6$ and $8.5 + 0.5 - 4 = 5$. Both planes are satisfied.

Note: three ways to describe a line

A line in three dimensions can be described by a vector equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$, by parametric equations $x = a_1 + \lambda d_1$, $y = a_2 + \lambda d_2$, $z = a_3 + \lambda d_3$, or by the symmetric Cartesian form $\dfrac{x - a_1}{d_1} = \dfrac{y - a_2}{d_2} = \dfrac{z - a_3}{d_3}$. Unlike in two dimensions, a line cannot be described by a single linear equation, because one equation in x, y and z describes a plane. The symmetric form is really two equations, which describe two planes whose intersection is the line. If a component of the direction vector is zero, the symmetric form needs adjusting: for $\mathbf{d} = (2, 0, 3)$ you would write $\dfrac{x - a_1}{2} = \dfrac{z - a_3}{3}$ with $y = a_2$.

Watch out: frequent mistakes

- Confusing the angle between a line and a plane with the angle between the line and the normal. The formula with the dot product gives $\sin$ of the angle to the plane, not $\cos$.
- Forgetting the modulus signs in the distance formulas, which gives a negative distance.
- Assuming that two lines which do not look parallel must intersect. In three dimensions they may be skew, so you must check all three coordinate equations.
- Using the cross product of vectors in the wrong order, which reverses the sign of the normal. This does not change the plane, but it can change the sign of an

intermediate result.
- Not checking that a point lies in the plane after finding the equation.

Applications of vectors in three dimensions

Vectors in three dimensions are the language of engineering and computer graphics. Architects use planes to describe roofs and floors, and the angle between two planes tells them how steeply a roof meets a wall. Air traffic controllers monitor the shortest distance between the lines representing flight paths, and they alert pilots when two aircraft will pass too close. Game designers use the dot product to decide whether a surface faces towards a light source, which determines how bright it should appear, and they use the cross product to compute the normal of every triangle in a model. In physics the moment of a force about a point, the force on a charged particle in a magnetic field, and the angular momentum of a spinning body are all given by cross products.

Method: revision plan for this chapter

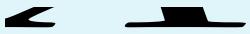
1. Practise computing dot and cross products until you can do them without hesitation, using the determinant layout for the cross product.
2. Learn the three formulae for distances: point to plane, point to line and between skew lines.
3. Practise converting between vector, parametric and Cartesian forms of lines and planes.
4. Always draw a quick sketch of the configuration, even if it is rough, to avoid mistakes in which side of a plane a point lies.

Thinking geometrically before calculating

Many mistakes in three-dimensional vector geometry come from rushing straight into algebra without a picture. Before you start, decide what kind of objects you are working with: a point, a line or a plane. A point has three coordinates. A line is described by a point and one direction. A plane is described by a point and two directions, or by a point and one normal direction. Next, ask what relationship the question is about. If it asks for an angle, you want the dot product. If it asks for a perpendicular direction or an area, you want the cross product. If it asks for a distance, you want one of the three distance formulae. Finally, ask whether the answer should be a point, a number or an equation, and give it in that form. This simple checklist turns an apparently complicated problem into a short sequence of familiar steps.

Worked example 7: a plane containing a line and a point

Find the equation of the plane that contains the line $\mathbf{r} = (1, 2, 0) + \lambda(1, 0, 1)$ and the point $B(0, 1, 3)$.

The plane contains the direction $\mathbf{d} = (1, 0, 1)$ of the line, and it also contains the vector from the point $A = (1, 2, 0)$ on the line to the point B, which is . The normal to the plane is the cross product of these two vectors:
$(1, 0, 1) \times (-1, -1, 3) = (0 \cdot 3 - 1 \cdot (-1),\ 1 \cdot (-1) - 1 \cdot 3,\ 1 \cdot (-1) - 0) = (1, -4, -1)$.
Using the point A, the right-hand side of the equation is $1 - 8 - 0 = -7$, so the plane is $x - 4y - z = -7$. Check the point B: $0 - 4 - 3 = -7$, correct. Check a second point on the line, $\lambda = 1$: $(2, 2, 1)$ gives $2 - 8 - 1 = -7$, also correct.

Watch out: skew lines and the intersection test

To decide whether two lines meet, write each in parametric form with different parameters, equate the three coordinates and solve two of the three equations for the parameters. Then substitute into the third equation as a check. If the third equation fails, the lines are skew, unless the direction vectors are parallel, in which case the lines are parallel or the same line. Never stop after solving only two equations, since two unknowns can always be found from two equations, even for lines that never meet. This is the single most common reason for losing marks in questions on intersecting lines, and the check takes only a few seconds.

Did you know?

The vector products used today were invented in the nineteenth century. William Rowan Hamilton, an Irish mathematician, discovered quaternions in 1843 while walking by the Royal Canal in Dublin, and he carved the formula for them into Broom Bridge. The cross product and dot product were extracted from the quaternions by Josiah Willard Gibbs and Oliver Heaviside around 1880, and they became the standard tools of physicists. Today the quaternions themselves are used inside every smartphone and video game, because they describe three-dimensional rotations without the problem of gimbal lock.

Otto's tip

Check any cross product by dotting it with each original vector. Both answers must be zero.