Worksheet - Further vectors

Year: Further Maths · 20

Show your working.

Question 1

Question 1 – The scalar product *(Section A · Skills practice)*

The vectors $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ are given.

a. Calculate $\mathbf{a} \cdot \mathbf{b}$, $|\mathbf{a}|$ and $|\mathbf{b}|$.
b. Find the angle between $\mathbf{a}$ and $\mathbf{b}$ to 1 decimal place.
c. Show that $3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ is perpendicular to $\mathbf{i} - \mathbf{j} + \mathbf{k}$.
d. Find the value of k for which $k\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is perpendicular to $3\mathbf{i} + k\mathbf{j} + 4\mathbf{k}$.

Question 2

Question 2 – The vector product *(Section A · Skills practice)*

Use the same vectors $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

a. Calculate $\mathbf{a} \times \mathbf{b}$ using a determinant.
b. Verify that $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.
c. Find the area of the parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$, and of the triangle with these two sides.

Question 3

Question 3 – Equations of a line *(Section A · Skills practice)*

A line passes through $A(1, 2, -1)$ and has direction vector $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$.

a. Write down the vector equation of the line.
b. Write the equation in Cartesian form.
c. A second line passes through $P(3, 0, 1)$ and $Q(5, -2, 4)$. Find a vector equation of line PQ.
d. Show that $(7, -4, 7)$ lies on PQ but $(1, 2, -1)$ does not.

Question 4

Question 4 – Equations of a plane *(Section A · Skills practice)*

Find the Cartesian equation of each plane.

a. The plane through $A(1, 0, 2)$ with normal vector $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$. Show that $(3, 1, 1)$ lies in the plane.
b. The plane through $A(1, 0, 0)$, $B(0, 2, 0)$ and $C(0, 0, 3)$.

c. Write the plane in part (a) in the form $\mathbf{r} \cdot \mathbf{n} = d$.

Question 5

Question 5 – Angles *(Section A · Skills practice)*

Find each angle to 1 decimal place.

a. The angle φ between the line $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$ and the plane $\mathbf{r} \cdot (2\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = 7$.
b. The angle between the planes $x + 2y + 2z = 5$ and $3x + 4z = 2$.

Question 6

Question 6 – Distance from a point to a plane *(Section A · Skills practice)*

The plane Π has equation $2x - 2y + z = 5$ and the point P is $(2, 3, -1)$.

a. Find the perpendicular distance from P to Π.
b. Write down the equation of the line through P perpendicular to Π.
c. Find the coordinates of the foot of the perpendicular and verify it lies on Π.

Question 7

Question 7 – A survey drone *(Section B · Real-life practice)*

A survey drone flies in a straight line. Relative to a mast at the origin O (distances in metres, z vertical), its position is , where t is time in seconds.

a. Find the speed of the drone.
b. Find the value of t when the drone is closest to the mast.
c. Find the shortest distance from the mast to the drone's path, to 1 decimal place.
d. The drone must stay at least 30 m from the mast. Does it?

Question 8

Question 8 – A sloping roof *(Section B · Real-life practice)*

The corners of a triangular roof panel are $P(0, 0, 4)$, $Q(6, 0, 4)$ and $R(0, 4, 6)$, where lengths are in metres and z is the height.

a. Find the vectors and and their cross product.
b. Show that the plane of the roof has equation $2z - y = 8$.
c. Find the area of the panel.
d. Find the angle between the roof and the horizontal ground, to 1 decimal place.

Question 9

Question 9 – A laser and a mirror *(Section B · Real-life practice)*

A laser beam is fired from the point $A(0, 0, 9)$ in the direction $\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ towards a flat mirror lying in the plane $2x + y + 2z = 12$ (lengths in metres).

a. Write the vector equation of the beam.
b. Find the position where the beam meets the mirror.
c. Find the angle between the beam and the mirror, to 1 decimal place.
d. What would happen if the beam were fired in the direction $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$?

Question 10

Question 10 – Two flight paths *(Section B · Real-life practice)*

Two aircraft fly along straight lines (units in km, z is altitude): l_1:

and l_2: .

a. Show that the lines do not intersect.
b. Find a vector perpendicular to both directions.
c. Find the shortest distance between the paths to 2 decimal places.
d. Air traffic control requires paths to be at least 3 km apart. Is this requirement met?

Question 11

Question 11 – Work done by a crane *(Section B · Real-life practice)*

A crane applies a constant force $\mathbf{F} = 4\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$ newtons to a load that moves through the displacement $\mathbf{s} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ metres.

a. Calculate the work done, $\mathbf{F} \cdot \mathbf{s}$.
b. Find the magnitudes $|\mathbf{F}|$ and $|\mathbf{s}|$.
c. Find the angle between the force and the displacement to the nearest degree.
d. Find the component of the force in the direction of the displacement.

Question 12

Question 12 – Tightening a bolt *(Section B · Real-life practice)*

A spanner applies a force $\mathbf{F} = (0, 30, -10)$ N at the point with position vector $\mathbf{r} = (0.4, 0, 0.1)$ m measured from the centre of a bolt. The moment (torque) is $\mathbf{r} \times \mathbf{F}$.

a. Calculate the moment vector.

b. Find its magnitude in N m.
c. Find the component of the moment about the z-axis, which is the axis of the bolt.
d. Explain why a longer spanner would make the bolt easier to turn.

Question 13

Question 13 – A tent shaped like a pyramid *(Section B · Real-life practice)*

A tent is a pyramid with its apex at the origin O and base vertices $A(2, 1, 0)$, $B(-1, 3, 2)$ and $C(0, -2, 4)$ (lengths in metres).

a. Find .

b. The volume of the tetrahedron $OABC$ is . Find it.

c. Find the area of the triangle OBC.
d. Hence find the perpendicular distance from A to the plane OBC.

Question 14

Question 14 – Two submarines *(Section B · Real-life practice)*

Two submarines move at constant velocities. Submarine A has position $\mathbf{r}_A = (1, 2, 3) + t(2, -1, 1)$ and submarine B has position $\mathbf{r}_B = (4, -1, 2) + t(-1, 2, 2)$, in kilometres, where t is the time in hours and z is height above the sea floor.

a. Show that the paths cross at the point $(3, 1, 4)$ and state the value of the parameter for each path.
b. Do the submarines collide? Explain.
c. Find the acute angle between the two paths, to 1 decimal place.
d. Find the speed of each submarine.

Question 15

Question 15 – Meeting, angle and reflection *(Section C · Problem solving)*

The line l is $\mathbf{r} = (1, 0, 2) + \lambda(1, 1, 1)$ and the plane Π is $x + 2y - z = 4$.

a. Find the coordinates of the point where l meets Π.
b. Find the acute angle between l and Π.
c. Show that the point $P(1, 0, 2)$ does not lie on Π and find its distance from Π.
d. Find the image of P when it is reflected in Π.

Question 16

Question 16 – Parallel lines *(Section C · Problem solving)*

The lines l_1: $\mathbf{r} = (1, 1, 0) + \lambda(1, 2, 1)$ and l_2: $\mathbf{r} = (2, -1, 3) + \mu(1, 2, 1)$ are parallel.

a. Find a vector joining a point on l_1 to a point on l_2.
b. Find the Cartesian equation of the plane containing both lines.
c. Find the shortest distance between the lines, in exact form.
d. Explain why the distance does not depend on which points you choose.

Question 17

Question 17 – The closest points on skew lines *(Section C · Problem solving)*

Use the lines l_1: $\mathbf{r} = (1, 2, 3) + \lambda(1, -1, 2)$ and l_2: $\mathbf{r} = (2, 0, -1) + \mu(2, 1, 1)$ from Question 10.

a. Write the vector from $A = (1 + \lambda, 2 - \lambda, 3 + 2\lambda)$ on l_1 to $B = (2 + 2\mu, \mu, -1 + \mu)$ on l_2.
b. Use and to find λ and μ.
c. Find the closest points A and B and the distance $|AB|$.
d. Verify the distance agrees with Question 10.

Question 18

Question 18 – A tetrahedron *(Section C · Problem solving)*

The points $A(2, 0, 0)$, $B(0, 3, 0)$ and $C(0, 0, 6)$ and the origin O form a tetrahedron.

a. Find the Cartesian equation of the plane ABC.
b. Find the volume of the tetrahedron.
c. Find the area of the face ABC.
d. Hence find the perpendicular distance from O to the plane ABC, and confirm it with the point-to-plane formula.

Question 19

Question 19 – From parametric to Cartesian *(Section C · Problem solving)*

The plane Π has vector equation .

a. Find a normal vector to Π.
b. Find the Cartesian equation of Π.
c. Show that $(3, 1, 1)$ lies in Π and find the values of λ and μ.
d. Find the distance of the origin from Π.

Question 20

Question 20 – Challenge: Lagrange's identity *(Section D · Challenge)*

Let $\mathbf{a}$ and $\mathbf{b}$ be vectors and let θ be the angle between them.

a. Show that $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$ using $\sin^2\theta = 1 - \cos^2\theta$ and the identity $|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$.
b. Verify the identity for $\mathbf{a} = (2, -1, 3)$ and $\mathbf{b} = (1, 4, -2)$.
c. Prove the identity for $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$ by expanding both sides.
d. Explain why $|\mathbf{a} \times \mathbf{b}|$ is the area of a parallelogram.