Answers - Further vectors

Year: Further Maths

Answer to Question 1

Method: the scalar product is $\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = |\mathbf{a}||\mathbf{b}|\cos\theta$. Two non-zero vectors are perpendicular exactly when their scalar product is 0.

a. Multiply corresponding components and add; use Pythagoras for the magnitudes. $\mathbf{a}\cdot\mathbf{b} = 2 - 4 - 6 = -8$; $|\mathbf{a}| = \sqrt{4+1+9} = \sqrt{14}$; $|\mathbf{b}| = \sqrt{1+16+4} = \sqrt{21}$

b. Rearrange the scalar product formula for $\cos\theta$.
$\cos\theta = \dfrac{-8}{\sqrt{14}\sqrt{21}} = \dfrac{-8}{\sqrt{294}} = -0.4666$, so $\theta = \mathbf{117.8}^\circ$ The angle is obtuse because the scalar product is negative.

c. Compute the scalar product. $3 - 1 - 2 = 0$, so the vectors are **perpendicular**. So the two vectors are at 90°.

d. Set the scalar product equal to zero and solve for k. $3k + 2k - 4 = 0$, so $k = \dfrac{4}{5}$
Check: $\frac{12}{5} + \frac{8}{5} - 4 = 0$.

Answer to Question 2

a. $-10\mathbf{i} + 7\mathbf{j} + 9\mathbf{k}$

b. $\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b}) = -20 - 7 + 27 = 0$ and $\mathbf{b}\cdot(\mathbf{a}\times\mathbf{b}) = -10 + 28 - 18 = 0$. ✓

c. Parallelogram: $|\mathbf{a}\times\mathbf{b}| = \sqrt{100+49+81} = \sqrt{230} \approx \mathbf{15.2}$. Triangle: half of this, **7.58**.

Answer to Question 3

a.

b. $\dfrac{x-1}{2} = \dfrac{y-2}{-1} = \dfrac{z+1}{3}$

c. Direction , so

d. For $(7,-4,7)$: $3 + 2\mu = 7$ gives $\mu = 2$; then $y = -4$ ✓ and $z = 1 + 6 = 7$ ✓. For $(1,2,-1)$: $3 + 2\mu = 1$ gives $\mu = -1$, but then $y = 2$ ✓ and $z = 1 - 3 = -2 \neq -1$, so it **does not** lie on the line.

Answer to Question 4

a. $\mathbf{r}\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n} = 2+6 = 8$, so $2x - y + 3z = 8$. For $(3,1,1)$: $6 - 1 + 3 = 8$ ✓.

b. and . Normal . Then $6x + 3y + 2z = 6\times 1 = 6$: $6x + 3y + 2z = 6$. (Check B: 6 ✓, C: 6 ✓.)

c.

Answer to Question 5

Method: the angle between a line with direction $\mathbf{d}$ and a plane with normal $\mathbf{n}$ satisfies $\sin\varphi = \dfrac{|\mathbf{d}\cdot\mathbf{n}|}{|\mathbf{d}||\mathbf{n}|}$ (the complement of the angle with the normal). The angle between two planes is the angle between their normals.

a. Here $\mathbf{d} = (1,2,2)$, $\mathbf{n} = (2,-1,2)$ and both have magnitude 3.
$\sin\varphi = \dfrac{|\mathbf{d}\cdot\mathbf{n}|}{|\mathbf{d}||\mathbf{n}|} = \dfrac{|2-2+4|}{3\times 3} = \dfrac{4}{9}$, so $\varphi = \mathbf{26.4}^\circ$ So the line meets the plane at about 26.4°.

b. The normals are $(1,2,2)$ with magnitude 3 and $(3,0,4)$ with magnitude 5. The angle between the planes equals the angle between their normals:
$\cos\theta = \dfrac{|(1,2,2)\cdot(3,0,4)|}{3\times 5} = \dfrac{11}{15}$, so $\theta = \mathbf{42.8}^\circ$ So the planes meet at about 42.8°.

Answer to Question 6

Method: the distance from the point (x_1,y_1,z_1) to the plane $ax+by+cz=d$ is $\dfrac{|ax_1+by_1+cz_1-d|}{\sqrt{a^2+b^2+c^2}}$. Alternatively, follow the normal line from the point until it meets the plane.

a. Substitute the coordinates of P into the formula.
$\dfrac{|2(2)-2(3)+(-1)-5|}{\sqrt{4+4+1}} = \dfrac{|-8|}{3} = \dfrac{8}{3}$ So P is $\frac{8}{3}\approx 2.67$ units from the plane.

b. A line perpendicular to the plane has the normal vector $(2,-2,1)$ as its direction.

The normal is the direction:

c. Substitute the general point of the line into the equation of the plane.
Substituting: $2(2+2t)-2(3-2t)+(-1+t)=5$ gives $4+4t-6+4t-1+t=5$, $9t=8$, $t=\frac{8}{9}$. The foot is $\left(\frac{34}{9},\frac{11}{9},-\frac{1}{9}\right)$. Check: $\frac{68}{9}-\frac{22}{9}-\frac{1}{9}=\frac{45}{9}=5$ ✓. Check: the distance from P to the foot is $\frac{8}{9}\times|(2,-2,1)| = \frac{8}{9}\times 3 = \frac{8}{3}$, as in part a).

Answer to Question 7

a. Speed $= |\mathbf{d}| = \sqrt{4 + 16 + 1} = \sqrt{21} \approx 4.58$ **m s**$^{-1}$.

b. The closest point has $\mathbf{r} \cdot \mathbf{d} = 0$: $(10 + 2t)2 + (-20 + 4t)4 + (30 - t)(-1) = 0$, $20 + 4t - 80 + 16t - 30 + t = 0$, $21t = 90$, $t = \frac{30}{7} \approx 4.29$ **s**.

c. $\mathbf{a} \times \mathbf{d} = (-100, 70, 80)$, magnitude $\sqrt{21300} = 145.9$. Distance $= \dfrac{145.9}{\sqrt{21}} =$ **31.8 m**.

d. $31.8 > 30$, so the drone **does stay at least 30 m away**.

Answer to Question 8

a. [redacted], [redacted], product $= (0 \cdot 2 - 0 \cdot 4,\ 0 \cdot 0 - 6 \cdot 2,\ 6 \cdot 4 - 0) = (0, -12, 24)$

b. The normal is $(0, -1, 2)$. $\mathbf{r} \cdot \mathbf{n} = (0, 0, 4) \cdot (0, -1, 2) = 8$, so $-y + 2z = 8$. Check Q: 8 ✓, R: $-4 + 12 = 8$ ✓.

c. Area $= \frac{1}{2}\sqrt{144 + 576} = \frac{1}{2}\sqrt{720} =$ **13.4 m**^{2}.

d. The horizontal has normal $\mathbf{k}$. $\cos\theta = \dfrac{|\mathbf{n} \cdot \mathbf{k}|}{|\mathbf{n}|} = \dfrac{2}{\sqrt{5}}$, so $\theta =$ **26.6**°.

Answer to Question 9

a. $\mathbf{r} = (0, 0, 9) + \lambda(1, 2, -3)$

b. Substitute: $2\lambda + 2\lambda + 2(9 - 3\lambda) = 12$, so $4\lambda + 18 - 6\lambda = 12$, $\lambda = 3$. The point is $(3, 6, 0)$.

c. $\sin\varphi = \dfrac{|(1, 2, -3) \cdot (2, 1, 2)|}{\sqrt{14} \times 3} = \dfrac{2}{3\sqrt{14}}$, so $\varphi =$ **10.3**° (a shallow angle).

d. $(1, 2, -2) \cdot (2, 1, 2) = 2 + 2 - 4 = 0$, so the beam is **parallel to the mirror** and never meets it (the point A is not on the mirror since $18 \neq 12$).

Answer to Question 10

a. Equating: $1 + \lambda = 2 + 2\mu$, $2 - \lambda = \mu$, $3 + 2\lambda = -1 + \mu$. From the second, $\mu = 2 - \lambda$; the first gives $1 + \lambda = 6 - 2\lambda$, $\lambda = \frac{5}{3}$, $\mu = \frac{1}{3}$. The third: $3 + \frac{10}{3} = \frac{19}{3}$ but $-1 + \frac{1}{3} = -\frac{2}{3}$. **Not equal**, so the lines do not meet; they are skew.

b. $\mathbf{d}_1 \times \mathbf{d}_2 = (-3, 3, 3)$, or $(-1, 1, 1)$.

c. [redacted] with A on l_1 and B on l_2.

[redacted] **4.04 km**.

d. $4.04 \geq 3$, so **yes**, the requirement is met.

Answer to Question 11

Method: the scalar product is $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = |\mathbf{a}||\mathbf{b}| \cos\theta$. Two non-zero vectors are perpendicular exactly when their scalar product is 0.

a. Work done is the scalar product of force and displacement. $12 - 2 + 10 =$ **20 J** So the crane does 20 J of work.

b. Use Pythagoras in three dimensions. $|\mathbf{F}| = \sqrt{16 + 4 + 25} = \sqrt{45} \approx 6.71$ N; $|\mathbf{s}| = \sqrt{9 + 1 + 4} = \sqrt{14} \approx 3.74$ m.

c. Use $\cos\theta = \dfrac{\mathbf{F} \cdot \mathbf{s}}{|\mathbf{F}||\mathbf{s}|}$. $\cos\theta = \dfrac{20}{\sqrt{45}\sqrt{14}} = \dfrac{20}{25.1} = 0.7968$, so $\theta =$ **37**° So the force acts at about 37° to the motion.

d. The component of $\mathbf{F}$ along $\mathbf{s}$ is $\mathbf{F} \cdot \hat{\mathbf{s}} = |\mathbf{F}| \cos\theta$. $\dfrac{\mathbf{F} \cdot \mathbf{s}}{|\mathbf{s}|} = \dfrac{20}{\sqrt{14}} =$ **5.35 N** Check: $5.35 \times \sqrt{14} = 20$, the work done.

Answer to Question 12

a. $\mathbf{r} \times \mathbf{F} = (0 \cdot (-10) - 0.1 \cdot 30,\ 0.1 \cdot 0 - 0.4 \cdot (-10),\ 0.4 \cdot 30 - 0) = (-3, 4, 12)$ **N m**

b. $\sqrt{9 + 16 + 144} = \sqrt{169} =$ **13 N m**

c. The z-component is **12 N m**, which turns the bolt.

d. The moment has magnitude $|\mathbf{r}||\mathbf{F}| \sin\theta$, so doubling $|\mathbf{r}|$ doubles the turning effect for the same force.

Answer to Question 13

Method: the vector product $\mathbf{a} \times \mathbf{b}$ is perpendicular to both vectors and has magnitude $|\mathbf{a}||\mathbf{b}| \sin\theta$, the area of the parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$. Compute it as a 3×3 determinant.

a. [redacted] $(16, 4, 2)$

b. This scalar triple product gives six times the volume of the tetrahedron. [redacted]. $V = \dfrac{36}{6} =$ **6 m**^{3}. So the tent holds 6 m³ of air.

c. The area of the triangle is half the magnitude of the vector product. $\frac{1}{2}\sqrt{256 + 16 + 4} = \frac{1}{2}\sqrt{276} =$ **8.31 m**^{2}.

d. Use the volume formula for a pyramid with base OBC. $V = \frac{1}{3} \times$ base $\times\, h$, so $h = \dfrac{3V}{\text{area}} = \dfrac{18}{8.31} =$ **2.17 m**. (Directly: $\dfrac{36}{\sqrt{276}} = 2.17$.) So A is about 2.17 m from

the plane OBC.

Answer to Question 14

a. For A: $1+2\lambda=3$, $2-\lambda=1$, $3+\lambda=4$ all give $\lambda=1$. For B: $4-\mu=3$, $-1+2\mu=1$, $2+2\mu=4$ all give $\mu=1$. So the paths cross at $(3,1,4)$.

b. Both submarines reach that point when $t=1$, because the parameters are times. So **yes, they collide** at $t=1$ hour.

c. $\cos\theta=\dfrac{|(2,-1,1)\cdot(-1,2,2)|}{\sqrt{6}\times 3}=\dfrac{|-2-2+2|}{3\sqrt{6}}=\dfrac{2}{3\sqrt{6}}$, so $\theta=$ **74.2°**

d. A: $\sqrt{4+1+1}=\sqrt{6}\approx$ **2.45 km h^{-1}**; B: $\sqrt{1+4+4}=$ **3 km h^{-1}**.

Answer to Question 15

a. $(1+\lambda)+2\lambda-(2+\lambda)=4$ gives $2\lambda-1=4$, $\lambda=\frac{5}{2}$. The point is $\left(\frac{7}{2},\frac{5}{2},\frac{9}{2}\right)$.

b. $\sin\varphi=\dfrac{|1+2-1|}{\sqrt{3}\sqrt{6}}=\dfrac{2}{\sqrt{18}}=0.4714$, so $\varphi=$ **28.1°**

c. At P: $1+0-2=-1\neq 4$. Distance $=\dfrac{|-1-4|}{\sqrt{6}}=\dfrac{5}{\sqrt{6}}\approx$ **2.04**.

d. The normal $(1,2,-1)$ points towards the plane (the value -5 is negative, so P is on the side where $x+2y-z$). The image is $P+2\cdot\dfrac{5}{6}(1,2,-1)=\left(\frac{8}{3},\frac{10}{3},\frac{1}{3}\right)$.

Check: the midpoint of P and its image is $\left(\frac{11}{6},\frac{5}{3},\frac{7}{6}\right)$, which satisfies $x+2y-z=\frac{11}{6}+\frac{20}{6}-\frac{7}{6}=4$. ✓

Answer to Question 16

a. $\mathbf{w}=(2,-1,3)-(1,1,0)=(1,-2,3)$

b. Normal $=\mathbf{d}\times\mathbf{w}=(1,2,1)\times(1,-2,3)=(8,-2,-4)$, or $(4,-1,-2)$. With the point $(1,1,0)$: $4-1=3$. So $4x-y-2z=3$. Check $(2,-1,3)$: $8+1-6=3$ ✓.

c. $\dfrac{|\mathbf{w}\times\mathbf{d}|}{|\mathbf{d}|}=\dfrac{\sqrt{64+4+16}}{\sqrt{6}}=\dfrac{\sqrt{84}}{\sqrt{6}}=\sqrt{14}\approx 3.74$

d. Sliding a point along a line changes $\mathbf{w}$ by a multiple of $\mathbf{d}$, and $\mathbf{d}\times\mathbf{d}=\mathbf{0}$, so the cross product, and hence the distance, is unchanged.

Answer to Question 17

a. 

b. $\mathbf{d}_1$: $(1+2\mu-\lambda)-(\lambda+\mu-2)+2(\mu-2\lambda-4)=0$ gives $-6\lambda+3\mu-5=0$. $\mathbf{d}_2$:

$2(1+2\mu-\lambda)+(\lambda+\mu-2)+(\mu-2\lambda-4)=0$ gives $-3\lambda+6\mu-4=0$. Solving: from the first $\mu=\dfrac{6\lambda+5}{3}$; substituting, $-3\lambda+12\lambda+10-4=0$, $\lambda=-\frac{2}{3}$ and $\mu=\frac{1}{3}$.

c. $A=\left(\frac{1}{3},\frac{8}{3},\frac{5}{3}\right)$, $B=\left(\frac{8}{3},\frac{1}{3},-\frac{2}{3}\right)$. and $|AB|=\frac{7}{3}\sqrt{3}=$ **4.04**.

d. $\dfrac{7\sqrt{3}}{3}=4.04$ matches $\dfrac{21}{\sqrt{27}}=\dfrac{21}{3\sqrt{3}}=\dfrac{7}{\sqrt{3}}$. ✓ Also , the common perpendicular.

Answer to Question 18

Method: a plane with normal $\mathbf{n}$ through the point A has equation $\mathbf{r}\cdot\mathbf{n}=\mathbf{a}\cdot\mathbf{n}$, that is $n_1x+n_2y+n_3z=d$. A normal to a plane containing two directions is their vector product.

a. Intercepts 2, 3, 6: $\dfrac{x}{2}+\dfrac{y}{3}+\dfrac{z}{6}=1$, or $3x+2y+z=6$. Check via .

b. $V=\frac{1}{6}\times 2\times 3\times 6=$ **6**.

c. **11.2**

d. $V=\frac{1}{3}\times 11.22\times h=6$ gives $h=\dfrac{18}{11.22}=1.60$. Formula: $\dfrac{|0-6|}{\sqrt{9+4+1}}=\dfrac{6}{\sqrt{14}}=1.60$. ✓

Answer to Question 19

Method: a plane with normal $\mathbf{n}$ through the point A has equation $\mathbf{r}\cdot\mathbf{n}=\mathbf{a}\cdot\mathbf{n}$, that is $n_1x+n_2y+n_3z=d$. A normal to a plane containing two directions is their vector product.

a. A normal is perpendicular to both direction vectors of the plane, so take their vector product. $(1,-1,1)$ Check:

$(1,-1,1)\cdot(1,1,0)=0$ and $(1,-1,1)\cdot(0,1,1)=0$.

b. Use the point $(1,0,2)$ to find d. $\mathbf{r}\cdot\mathbf{n}=(1,0,2)\cdot(1,-1,1)=3$: $x-y+z=3$

c. Substitute into the Cartesian equation, then compare components with the vector equation. $3-1+1=3$ ✓. Equate: $1+\lambda=3$ gives $\lambda=2$; $\lambda+\mu=1$ gives $\mu=-1$; $2+\mu=1$ ✓. So $\lambda=2$, $\mu=-1$. All three equations are consistent, so the point lies in the plane.

d. Use: the distance from the point (x_1,y_1,z_1) to the plane $ax+by+cz=d$ is

$$\frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}} \cdot \frac{|0 - 3|}{\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3} \approx 1.73$$

Answer to Question 20

a. Dividing by $|\mathbf{a}|^2|\mathbf{b}|^2$: $\frac{|\mathbf{a} \times \mathbf{b}|^2}{|\mathbf{a}|^2|\mathbf{b}|^2} = 1 - \cos^2\theta = \sin^2\theta$. Taking positive square roots (since $0 \leq \theta \leq \pi$) gives $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$.

b. $|\mathbf{a} \times \mathbf{b}|^2 = 230$. $|\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 = 14 \times 21 - 64 = 294 - 64 = 230$. ✓

c. The left side is $(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2$. Expanding gives the sum of terms $a_i^2b_j^2$ with $i \neq j$, minus $2a_ia_jb_ib_j$ over the three pairs. The right side is $(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$, and the terms $a_i^2b_i^2$ cancel, leaving the same expression. So the two sides are equal.

d. A parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$ has base $|\mathbf{a}|$ and height $|\mathbf{b}| \sin\theta$, so its area is $|\mathbf{a}||\mathbf{b}| \sin\theta = |\mathbf{a} \times \mathbf{b}|$.