Test answers - Further calculus

Year: Further Maths · Duration: 60 minutes · /20

Answer to Question 1 (4 marks)

Method: an improper integral has an infinite limit or an integrand that is unbounded at an end point. Replace the problem limit by a letter (R or a), integrate normally, then let the letter tend to its limit. If the limit is finite, the integral converges. Mark scheme as shown in brackets.

a. Replace ∞ by R and integrate: $\int 6e^{-2x}dx = -3e^{-2x}$. $\int_0^R 6e^{-2x}dx = 3\left(1-e^{-2R}\right)$ [1] $\to$ **3** [1] So the integral converges to 3.

b. Replace ∞ by R and integrate $x^{-1/2}$. $\int_1^R x^{-1/2}dx = 2\sqrt{R}-2$ [1], which tends to infinity as $R \to \infty$, so the integral **diverges** [1]

Answer to Question 2 (5 marks)

Method: use $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a}+c$ and $\displaystyle\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a}+c$. If the quadratic is not in this form, complete the square first and then substitute $u = x+k$. Mark scheme as shown in brackets.

a. Complete the square, then use the standard result with $a=1$ and $u=x+1$. $x^2+2x+2=(x+1)^2+1$ [1]. $[\arctan(x+1)]_0^1$ [1]
$= \arctan 2 - \arctan 1 = \arctan 2 - \dfrac{\pi}{4}$ [1] Numerically this is about 0.322.

b. Here $a=2$. $\left[\arcsin\dfrac{x}{2}\right]_0^1$ [1] $= \arcsin\dfrac{1}{2} = \dfrac{\pi}{6}$ [1]

Answer to Question 3 (5 marks)

a. $V = \pi\displaystyle\int_0^1 \frac{1}{1+x^2}dx$ [1] $= \pi[\arctan x]_0^1$ [1] $= \pi\cdot\dfrac{\pi}{4} = \dfrac{\pi^2}{4}$ [1]

b. At $x=1$, $y^2=\frac{1}{2}$, so the area is $\dfrac{\pi}{2}$ [1]. A cylinder has volume $\dfrac{\pi}{2}\approx 1.57$, which is less than $V\approx 2.47$, since the radius is larger for $x<1$ [1]

Method: for rotation about the x-axis, $V=\pi\int y^2\,dx$, and here $y^2=\dfrac{1}{1+x^2}$, a standard arctan integral. Common errors: forgetting the factor π, or integrating y instead of y^2. In (b), the comparison works because y decreases from 1 at $x=0$ to $\frac{1}{\sqrt{2}}$ at $x=1$, so the cylinder fits inside the solid.

Answer to Question 4 (3 marks)

Method: the mean value of f on $[a, b]$ is $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$: the height of the rectangle on $[a, b]$ with the same area as the region under the curve. Mark scheme as shown in brackets.

a. $\int_0^4 (4x - x^2)dx = \left[2x^2 - \dfrac{x^3}{3}\right]_0^4$ [1] $= 32 - \dfrac{64}{3} = \dfrac{32}{3}$ [1]. Mean $= \dfrac{1}{4} \cdot \dfrac{32}{3} = \dfrac{8}{3}$ [1]

Answer: the mean value is $\frac{8}{3} \approx 2.67$, which is $\frac{2}{3}$ of the maximum value 4 of the parabola.

By symmetry about $x = 2$, the same mean value is obtained on $[0, 2]$.

Answer to Question 5 (3 marks)

Method: rotating about the x-axis gives $V = \pi\displaystyle\int_a^b y^2\,dx$; about the y-axis, $V = \pi\displaystyle\int_c^d x^2\,dy$, with the limits in y. Square the function first, then integrate. Mark scheme as shown in brackets.

a. The top of the planter is where $y = 9$. $9 = \dfrac{x^2}{4}$, $x = 6$: **6 cm** [1]

b. Rotating about the y-axis, write $x^2 = 4y$ and integrate with respect to y from 0 to 9. $x^2 = 4y$, $V = \pi\displaystyle\int_0^9 4y\,dy$ [1] $= \pi \cdot 2 \cdot 81 = 162\pi \approx$ **509 cm**^{3} [1] So the planter holds about 509 cm³ of compost.