Lesson - Further calculus

Year: Further Maths

This chapter extends integration: improper integrals defined by limits, integrals that give inverse trigonometric functions, volumes of revolution and the mean value of a function.

Starter activity

A rainwater tank is being filled by a leaking pipe, and the rate at which water enters falls by half every hour: first 100 litres in the first hour, then 50, then 25, and so on. Does the tank fill up with an infinite amount of water, or does the total settle at a limit? Add the first six hours by hand. Then compare with the integral $\int_0^{\infty} 100e^{-0.69t}\,dt$. What do you notice about the answer, and what does it say about adding infinitely many things that get smaller and smaller?

Improper integrals

Definition: improper integral

An integral is **improper** if a limit of integration is infinite, or the integrand is undefined at a point in the range. It is defined as a limit. If the limit exists the integral **converges** to that value; otherwise it **diverges**.

Rule: limits

Infinite limit: $\displaystyle\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^{b} f(x)\,dx.$

Undefined at a: $\displaystyle\int_a^{b} f(x)\,dx = \lim_{\varepsilon\to 0^+}\int_{a+\varepsilon}^{b} f(x)\,dx.$

Key fact: powers of x

$\int_1^{\infty} x^{-p}\,dx$ converges if and only if $p > 1$. $\int_0^{1} x^{-p}\,dx$ converges if and only if p. Also $x\ln x \to 0$ and $xe^{-x} \to 0$ in the relevant limits.

Method: evaluating an improper integral

1. Say which type it is and replace the problem limit by b or ε.
2. Integrate as normal.
3. Substitute the limits, then take $b \to \infty$ or $\varepsilon \to 0$.
4. State "converges to ..." or "diverges".

Example 1

Evaluate $\int_0^1 \frac{1}{\sqrt{x}}dx$**.**

$\lim_{\varepsilon\to 0}\left[2\sqrt{x}\right]_{\varepsilon}^{1} = 2 - 0 = 2.$

Example 2

Evaluate $\int_0^1 \ln x\,dx$ **and** $\int_1^\infty \frac{1}{x}dx$**.**

First: $\lim_{\varepsilon\to 0}\left[x\ln x - x\right]_{\varepsilon}^{1} = -1 - 0 = -1$. Second: $\lim_{b\to\infty}\ln b$ does not exist, so it diverges.

Example 3: partial fractions

Evaluate $\int_1^\infty \frac{dx}{x(x+1)}$**.**

$\int_1^\infty \left(\frac{1}{x} - \frac{1}{x+1}\right)dx = \lim_{b\to\infty}\left[\ln\frac{x}{x+1}\right]_1^b = 0 - \ln\frac{1}{2} = \ln 2.$

Watch out

Never substitute ∞ directly. Always show the limit.

Inverse trigonometric integrals

Rule: standard results

$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$, $\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$, $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$. Hence

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin\frac{x}{a} + c, \qquad \int \frac{1}{a^2+x^2}dx = \frac{1}{a}\arctan\frac{x}{a} + c.$$

The constant a must match the number under the root or in the denominator.

Method: quadratics under a root or in a denominator

1. Complete the square, for example $x^2 + 4x + 13 = (x+2)^2 + 9$.
2. Substitute $u = x + 2$.
3. Quote the standard result with the correct a.
4. Use partial fractions first if the denominator factorises.

Example 4

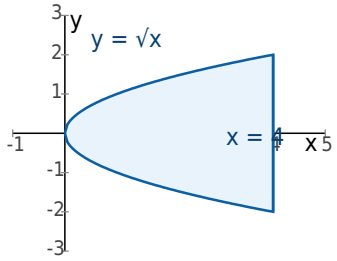
Find $\int \dfrac{dx}{x^2+4x+13}$.

$x^2+4x+13=(x+2)^2+9$, so the integral is $\frac{1}{3}\arctan\dfrac{x+2}{3}+c$.

Volumes of revolution

Rule: volume of revolution

Rotating the region under $y=f(x)$, $a\le x\le b$, through 2π about the x-axis: $V=\pi\displaystyle\int_a^b y^2\,dx$. About the y-axis, $c\le y\le d$: $V=\pi\displaystyle\int_c^d x^2\,dy$, after writing x^2 in terms of y. Between two curves $f\ge g$: $V=\pi\int(f^2-g^2)dx$, not $(f-g)^2$. Parametric: $V=\pi\int y^2\frac{dx}{dt}dt$.



Rotating the region under y = √x (0 ≤ x ≤ 4) about the x-axis gives a solid; each slice is a disc of radius √x.

Example 5

The region under $y=\sqrt{x}$, $0\le x\le 4$, is rotated about the x-axis. Find the volume.

$V=\pi\int_0^4 x\,dx=\pi\left[\frac{x^2}{2}\right]_0^4=8\pi$.

Example 6: about the y-axis

The region between $y=x^2$ and the y-axis, for $0\le y\le 4$, is rotated about the y-axis. Find the volume.

$x^2=y$, so $V=\pi\int_0^4 y\,dy=\pi\left[\frac{y^2}{2}\right]_0^4=8\pi$.

Note: standard solids

A cone of radius r and height h has $V=\frac{1}{3}\pi r^2h$; a sphere of radius R has $V=\frac{4}{3}\pi R^3$.

Both come from this formula and make good checks.

Mean value of a function

Definition: mean value

The mean value of $f(x)$ on $[a, b]$ is $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$.

Example 7

Find the mean value of x^2 on $[0, 3]$, and of $\dfrac{1}{x}$ on $[1, e]$.

$\frac{1}{3}\left[\frac{x^3}{3}\right]_0^3 = 3$. And $\dfrac{1}{e-1}\int_1^e \dfrac{dx}{x} = \dfrac{1}{e-1} \approx 0.582$.

Watch out

- Forgetting the factor π, or integrating y instead of y^2.
- Using $(f-g)^2$ for the region between two curves.
- Dividing by the wrong interval length for a mean value.
- Mixing x and dy in a rotation about the y-axis.

Note: two kinds of improper integral

An integral is improper for one of two reasons. Either one of the limits of integration is infinite, as in $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$, or the integrand becomes infinite at a point of the interval, as in $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$ where the function is unbounded at $x = 0$. In the second case the unbounded point might be in the middle of the interval, in which case you must split the integral there and treat each side separately. The integral converges only if both pieces converge. A careless calculation that simply substitutes limits into an antiderivative may give an answer that looks perfectly sensible but is completely wrong, as in the integral of $\dfrac{1}{(x-1)^2}$ from 0 to 2 where the formula gives -2 even though the integrand is positive.

Method: evaluating an improper integral

1. Identify the problem: an infinite limit, or a point where the integrand is undefined.
2. Replace the problem by a finite value, such as R or a, and integrate normally.

3. Take the limit as $R \to \infty$ or $a \to 0^+$, using standard limits such as $e^{-R} \to 0$ and $a \ln a \to 0$.
4. State clearly whether the limit exists. If it does the integral converges to that value; otherwise it diverges.

Worked example 5: an integral that needs integration by parts

Show that $\displaystyle\int_0^{\infty} x^2 e^{-x}\,dx = 2$.

Replace the upper limit by R. Integrate by parts with $u = x^2$ and $dv = e^{-x}dx$, which gives $-x^2e^{-x} + 2\int xe^{-x}dx$. A second application of parts gives $\int xe^{-x}dx = -xe^{-x} - e^{-x}$. Putting these together, an antiderivative is $-e^{-x}\left(x^2 + 2x + 2\right)$. Between 0 and R this gives $2 - e^{-R}\left(R^2 + 2R + 2\right)$. As $R \to \infty$ the exponential decay beats the polynomial growth, so the second term tends to zero and the integral is 2. It is important to say why the boundary term vanishes, because the mark scheme gives a mark for the justification.

Key fact: standard inverse trigonometric integrals

For a positive constant a, $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} + C$ for $|x|$, and $\displaystyle\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\arctan\frac{x}{a} + C$. These come from differentiating the inverse functions: if $y = \arcsin x$ then $\sin y = x$, so $\cos y\dfrac{dy}{dx} = 1$ and $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1 - x^2}}$. The formulae are given in the formula booklet, but you must be able to adapt them. If the quadratic in the denominator is not already in the right form, complete the square first, and if the coefficient of x^2 is not 1, factor it out.

Worked example 6: a substitution

Use the substitution $x = 3\sin\theta$ to show that $\displaystyle\int_0^3 \sqrt{9 - x^2}\,dx = \frac{9\pi}{4}$.

With $x = 3\sin\theta$ we have $dx = 3\cos\theta\,d\theta$ and $\sqrt{9 - x^2} = 3\cos\theta$. The limits $x = 0$ and $x = 3$ become $\theta = 0$ and $\theta = \frac{\pi}{2}$. The integral becomes $\displaystyle\int_0^{\pi/2} 9\cos^2\theta\,d\theta = \frac{9}{2}\int_0^{\pi/2}(1 + \cos 2\theta)\,d\theta = \frac{9}{2}\cdot\frac{\pi}{2} = \frac{9\pi}{4}$. As a check, this is a quarter of a circle of radius 3, whose area is $\frac{1}{4}\pi \cdot 9$. The geometrical check agrees with the calculation.

Method: volume of revolution

1. Sketch the region and decide whether it is rotated about the x-axis or the y-axis.

2. About the x-axis, the cross-section at x is a disc of radius y, so $V = \pi \int_a^b y^2 \, dx$. About the y-axis, write x in terms of y and use $V = \pi \int_c^d x^2 \, dy$.
3. If the region lies between two curves $y = f(x)$ above $y = g(x)$, subtract the volumes: $V = \pi \int_a^b \left(f^2 - g^2\right) dx$. Do not square the difference $f - g$.
4. Keep the answer in terms of π unless a decimal is requested, and give units for volumes.

Watch out: errors with volumes

- Squaring the difference of two functions instead of subtracting their squares: $(f - g)^2 \neq f^2 - g^2$.
- Rotating about the y-axis but integrating with respect to x. If you use $\pi \int x^2 \, dy$, the limits must be y-values.
- Forgetting the factor π, or forgetting to square the radius.
- Using the wrong limits when two curves intersect: always solve for the intersection first.

Worked example 7: mean value in context

The speed of a train in km per hour is $v = 60 \sin \dfrac{\pi t}{2}$ for the first two hours, with t in hours. Find its mean speed.

The mean value of v over the interval from 0 to 2 is $\dfrac{1}{2} \int_0^2 60 \sin \dfrac{\pi t}{2} \, dt$. The integral is $60 \cdot \dfrac{2}{\pi} \left[-\cos \dfrac{\pi t}{2} \right]_0^2 = \dfrac{120}{\pi}(1 + 1) = \dfrac{240}{\pi}$. Dividing by the length of the interval gives a mean speed of $\dfrac{120}{\pi} \approx 38.2$ km per hour. Notice that the mean is less than the maximum of 60, as it must be, and that the total distance travelled is the integral itself, about 76.4 km.

Applications of further calculus

Improper integrals are used wherever a process continues for ever, such as the total energy radiated by a decaying source, the total dose of a drug that is eliminated slowly, or the area under the normal curve in statistics. Volumes of revolution are used by engineers to find the capacity of tanks, bowls, funnels and bottles, and by designers to calculate the amount of material needed for turned objects such as table legs. The mean value of a function appears in electrical engineering, where the effective value of an alternating current is found from the mean of the square of the current. In all of these cases, the same ideas apply: an integral adds up a very large number of small pieces, and the exact answer is found by taking a limit.

Method: revision plan for this chapter

1. Memorise the two inverse trigonometric integrals and practise completing the square.
2. Work through improper integrals with infinite limits first, then those with a point of discontinuity.
3. Draw a sketch for every volume question and label the axis of rotation.
4. Use the test for convergence of $\int_1^\infty x^{-p}dx$ to decide quickly whether a simple integral converges.
5. Check answers by estimating the area or volume with a rough sketch.

Why limits are needed

Students sometimes ask why we cannot simply substitute infinity into an antiderivative. The reason is that infinity is not a number, and arithmetic with it is not reliable. The limit gives the idea a precise meaning. When we write the improper integral as the limit of ordinary integrals, we are asking a clear question: as the upper limit is pushed further and further out, do the areas settle down to a fixed number, or do they keep growing? If the areas settle down, the integral converges and the fixed number is its value. If they keep growing, or keep oscillating without settling, the integral diverges. This is the same idea as the sum of an infinite geometric series, where the partial sums approach a limit when the common ratio is small enough. Seeing the connection between series and integrals is useful, because the same tests of convergence appear in both.

Comparing rates of decay

It is useful to have a feel for which functions decay fast enough to have a finite integral. Exponential decay, such as e^{-x}, is so rapid that the area to infinity is always finite. Powers of x are borderline: the function $\dfrac{1}{x^2}$ decays fast enough, giving a finite area, but $\dfrac{1}{x}$ and $\dfrac{1}{\sqrt{x}}$ do not. The point $p = 1$ is the dividing line. Near zero the situation is reversed, because x^{-p} blows up there. A function that is large close to zero can only have a finite integral if the power is less than one. When you meet a new integral, ask yourself how the integrand behaves at the awkward end, and compare it with a power that you know. This habit of comparison will serve you well in later courses on series and analysis.

Worked example 8: using an inverse trigonometric result in a problem

The speed of a particle is given by $v = \dfrac{6}{9 + t^2}$ metres per second for $t \geq 0$.

Find the total distance travelled as $t \to \infty$, and the time at which the particle has travelled half of this distance.

The distance travelled is the integral of the speed. Using the standard result with $a = 3$, the integral of $\dfrac{6}{9 + t^2}$ is $6 \cdot \dfrac{1}{3} \arctan \dfrac{t}{3} = 2 \arctan \dfrac{t}{3}$. As $t \to \infty$, the arctangent tends to $\frac{\pi}{2}$, so the total distance is π metres, about 3.14 m. For half of this distance we need $2 \arctan \dfrac{T}{3} = \dfrac{\pi}{2}$, so $\arctan \dfrac{T}{3} = \dfrac{\pi}{4}$, which means $\dfrac{T}{3} = 1$ and $T = 3$ seconds. The particle covers half of its eventual distance in only three seconds, and then creeps along the rest of the way very slowly.

Shape of the solid and the choice of method

A good habit before any volume question is to picture the solid. A region under a curve and above the axis of rotation produces a solid with no hole, and the cross-section is a full disc. A region that is separated from the axis of rotation produces a solid with a hole, and each cross-section is a ring or washer, whose area is the area of the outer disc minus the area of the inner disc. That is exactly why the formula for a region between two curves subtracts the squares. If the region is rotated about the y-axis and the equation is given as $y = f(x)$, then you must first rearrange to express x^2 in terms of y, and replace the limits by the corresponding values of y. If the rearrangement is awkward, consider whether the question has been set so that the other axis is easier.

Watch out: leaving answers in the wrong form

Examiners usually ask for exact answers, and for volumes this means leaving the answer in terms of π and sometimes e or a logarithm. Giving a decimal approximation loses the accuracy mark. Conversely, when a question is set in context, such as the capacity of a bowl, the answer should be given as a decimal with units, correct to a sensible number of significant figures. Read the final line of the question carefully and decide which form is wanted. For improper integrals always write the conclusion in words, such as "the integral converges to 2" or "the integral diverges", because the conclusion carries a mark.

Worked example 9: a region bounded by an exponential

The region R is bounded by the curve $y = e^{-x}$, the axes and the line $x = \ln 3$. Find the area of R, the volume when R is rotated about the x-axis, and the mean value of y over the interval.

The area is $\int_0^{\ln 3} e^{-x} dx = 1 - e^{-\ln 3} = 1 - \frac{1}{3} = \frac{2}{3}$. For the volume, $y^2 = e^{-2x}$, so $V = \pi \int_0^{\ln 3} e^{-2x} dx = \frac{\pi}{2}\left(1 - e^{-2\ln 3}\right) = \frac{\pi}{2}\left(1 - \frac{1}{9}\right) = \frac{4\pi}{9}$. The mean value of y is the area divided by the length of the interval, so it is $\dfrac{2/3}{\ln 3} = \dfrac{2}{3 \ln 3} \approx 0.607$. Notice how $e^{-\ln 3}$ simplifies to $\frac{1}{3}$, a useful trick that appears often when the upper limit is a logarithm.

Link to earlier topics

Everything in this chapter builds on integration techniques that you already know: substitution, integration by parts and partial fractions. The new element is that the limits or the integrand are extreme, so you must combine a familiar technique with a limit. It is therefore good practice to separate the work into two clear stages. First find the antiderivative using the ordinary methods, and check it by differentiating mentally. Then deal with the limit, quoting any standard results such as the fact that powers of x grow more slowly than e^x. Students who mix the two stages tend to make slips, but those who write each stage separately find the questions straightforward. Finally, read the whole question again to make sure you have answered what was asked, using the right units and the right level of accuracy.

Estimating to check an answer

You can often check a volume or a mean value with a quick estimate. For a mean value, the answer must lie between the smallest and the largest value of the function on the interval. For a volume of revolution, compare it with a cylinder or cone that fits around the solid: the volume must be smaller than that of the cylinder through the widest point and larger than that of a cylinder through the narrowest point. These checks take only a few seconds and catch most arithmetic mistakes, such as a missing factor of π or a mis-squared radius.

Did you know?

The surprising idea that an infinitely long shape can have a finite volume was studied by the Italian mathematician Evangelista Torricelli in 1641, working before calculus was invented. His horn, now called Gabriel's horn, shocked philosophers of the time, because it seemed to say that infinity could be contained. The modern definition of an improper integral through limits, which resolves the paradox, was only made precise in the nineteenth century by Augustin-Louis Cauchy.

Otto's tip

Picture each slice as a thin disc of radius y and thickness dx: its volume is $\pi y^2 dx$. Then integrate the slices.