Worksheet - Further calculus

Year: Further Maths · 20

Show your working.

Question 1

Question 1 – Basic improper integrals *(Section A · Skills practice)*

Evaluate each improper integral, or show that it diverges. Define each one as a limit.

a. $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx$

b. $\displaystyle\int_1^{\infty} \frac{1}{x}\,dx$

c. $\displaystyle\int_0^{1} \frac{1}{\sqrt{x}}\,dx$

d. $\displaystyle\int_0^{\infty} e^{-3x}\,dx$

Question 2

Question 2 – Integration by parts and partial fractions *(Section A · Skills practice)*

Evaluate each improper integral and state clearly whether it converges.

a. $\displaystyle\int_0^{\infty} xe^{-x}\,dx$

b. $\displaystyle\int_0^{1} \ln x\,dx$

c. $\displaystyle\int_0^{2} \frac{1}{(x-1)^2}\,dx$ (look carefully at the integrand)

d. $\displaystyle\int_1^{\infty} \frac{1}{x(x+1)}\,dx$

Question 3

Question 3 – Inverse trigonometric integrals *(Section A · Skills practice)*

Use the standard results $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a}$ and $\displaystyle\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a}$.

a. Find $\displaystyle\int \frac{dx}{\sqrt{9-x^2}}$.

b. Find $\int_0^1 \frac{dx}{1+4x^2}$, giving the answer as an exact value and to 4 decimal places.

c. Evaluate $\int_0^{\sqrt{2}} \frac{dx}{\sqrt{4-x^2}}$ exactly.

d. Complete the square to evaluate $\int_{-1}^{1} \frac{dx}{x^2+2x+5}$.

Question 4

Question 4 – Volumes of revolution *(Section A · Skills practice)*

Find the exact volume generated when each region is rotated through 360° about the axis stated. Use $V = \pi \int y^2\,dx$ (about the x-axis) or $V = \pi \int x^2\,dy$ (about the y-axis).

a. The region under $y = \sqrt{x}$ from $x = 0$ to $x = 4$, about the x-axis.
b. The region under $y = x^2$ from $x = 0$ to $x = 2$, about the x-axis.
c. The region between $y = x^2$, the y-axis and $y = 4$, about the y-axis.
d. The region under $y = e^x$ from $x = 0$ to $x = 1$, about the x-axis.

Question 5

Question 5 – Region between two curves *(Section A · Skills practice)*

The region R is bounded by the line $y = x$ and the curve $y = x^2$, for $0 \le x \le 1$.

a. Sketch the region and find the x-coordinates of the points where the line and the curve meet.
b. Explain why the volume formed when R is rotated about the x-axis is $\pi \int_0^1 \left(x^2 - x^4\right)dx$.
c. Evaluate the volume exactly.

Question 6

Question 6 – Mean value of a function *(Section A · Skills practice)*

The mean value of $f(x)$ over the interval $[a, b]$ is $\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$. Find the mean value of each function on the interval given.

a. $f(x) = \sin^2 x$ on $[0, \pi]$
b. $f(x) = x^2$ on $[1, 3]$
c. $f(x) = e^{-x}$ on $[0, 2]$, to 4 decimal places
d. $f(x) = \frac{1}{1+x^2}$ on $[0, 1]$

Question 7

Question 7 – Absorbing a drug *(Section B · Real-life practice)*

After a patient swallows a tablet, the drug is absorbed into the blood at the rate $r(t) = 40e^{-0.5t}$ milligrams per hour, where t is the time in hours after taking the tablet.

a. Write an integral for the total mass of drug absorbed in the first T hours.
b. Evaluate it, and find the mass absorbed in the first 2 hours.
c. Explain why the total mass absorbed over all time is given by an improper integral, and find its value.
d. What percentage of the total is absorbed in the first 2 hours?

Question 8

Question 8 – The soup bowl *(Section B · Real-life practice)*

A designer models the inside of a soup bowl by rotating the curve $y = x^2$ through 360° about the y-axis, where x and y are in centimetres and the bowl is 8 cm deep.

a. Show that the volume of soup when the bowl is filled to a depth h cm is $\dfrac{\pi h^2}{2}$ cm^3.
b. Find the capacity of the bowl in cm^3 to the nearest cm^3.
c. Find the radius of the surface of the soup when the depth is 6 cm.
d. Soup is poured in at 20 cm^3 per second. How long does it take to reach a depth of 6 cm?

Question 9

Question 9 – An ice cream cone *(Section B · Real-life practice)*

An ice cream cone is made by rotating the line $y = \dfrac{4}{9}x$ about the x-axis for $0 \le x \le 9$, where lengths are in centimetres.

a. Find the radius of the open end and the height of the cone.
b. Use integration to show that the volume is 48π cm^3.
c. Check your answer with the formula $\frac{1}{3}\pi r^2 h$.
d. Ice cream is 1.2 times denser than the water in a drink of the same volume. Find the mass of ice cream that fills the cone, taking water as 1 g per cm^3.

Question 10

Question 10 – The observatory dome *(Section B · Real-life practice)*

The roof of an observatory is a hemisphere, formed by rotating the curve $y = \sqrt{100 - x^2}$ for $0 \le x \le 10$ about the x-axis (units are metres).

a. Show that the curve is part of a circle and state its centre and radius.
b. Use integration to find the volume under the dome.
c. Check with the formula for a hemisphere, $\frac{2}{3}\pi r^3$.
d. The heating system needs 0.08 kW for every cubic metre of air. Find the power needed to heat the air under the dome.

Question 11

Question 11 – Average temperature *(Section B · Real-life practice)*

Between 06:00 and 18:00 the temperature in a greenhouse, in °C, is modelled by $T = 14 + 6\sin\dfrac{\pi t}{12}$, where t is the number of hours after 06:00.

a. Find the temperature at 06:00, 12:00 and 18:00.
b. Write down the integral that gives the mean temperature over the 12 hours.
c. Evaluate the integral to find the mean temperature to 3 significant figures.
d. Explain why the mean is greater than the temperature at 06:00 but less than the maximum.

Question 12

Question 12 – Electrical power *(Section B · Real-life practice)*

The power delivered to a lamp by an alternating current is $P = 100\sin^2(120\pi t)$ watts, where t is time in seconds.

a. Show that $P = 50 - 50\cos(240\pi t)$.
b. Find the period of P.
c. Find the mean power over one period by integration.
d. Find the energy delivered in one minute.

Question 13

Question 13 – Charge from a current *(Section B · Real-life practice)*

When a switch is closed, the current in a circuit is $i(t) = 3e^{-2t}$ amperes for $t \geq 0$ in seconds.

a. Find the charge that flows in the first second, given that charge is the integral of current.
b. Find the total charge that flows after the switch is closed.
c. After what time has half the total charge flowed?
d. Find the mean current over the first second.

Question 14

Question 14 – A probability density *(Section B · Real-life practice)*

The time between two arrivals at an unusual switchboard has probability density $f(x) = \dfrac{1}{\pi(1+x^2)}$ for $-\infty$ (this is the Cauchy distribution).

a. Show that f is a valid probability density by evaluating $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$.
b. Find $P(0$.
c. Find $P(X > 1)$.
d. Explain why the mean $\displaystyle\int_{-\infty}^{\infty} x f(x)\,dx$ does not exist.

Question 15

Question 15 – When does an improper integral converge? *(Section C · Problem solving)*

This problem explores the integrals $\displaystyle\int_1^{\infty} x^{-p}\,dx$ and $\displaystyle\int_0^{1} x^{-p}\,dx$.

a. Show that $\displaystyle\int_1^{\infty} x^{-p}\,dx = \frac{1}{p-1}$ for $p > 1$ and that the integral diverges for $p \le 1$.
b. Find the value of p such that $\displaystyle\int_1^{\infty} x^{-p}\,dx = \tfrac{1}{2}$.
c. Show that $\displaystyle\int_0^{1} x^{-p}\,dx$ converges to $\dfrac{1}{1-p}$ if p and diverges otherwise.
d. Use the substitution $x = u^2$ to evaluate $\displaystyle\int_0^{\infty} \frac{dx}{\sqrt{x}(1+x)}$.

Question 16

Question 16 – A solid between a line and a parabola *(Section C · Problem solving)*

The curve C has equation $y = x^2$ and the line L has equation $y = 2x$. The region R lies between C and L.

a. Find the coordinates of the points where L meets C.
b. Find the area of R.
c. Find the exact volume formed when R is rotated through 360° about the x-axis.
d. A lampshade is modelled by this solid with lengths in decimetres. Convert the volume to litres to 3 significant figures (1 dm^3 = 1 litre).

Question 17

Question 17 – Designing a vase *(Section C · Problem solving)*

A glass vase is formed by rotating the curve $y = 1 + \sin x$ for $0 \le x \le \pi$ about the x-axis (lengths in centimetres).

a. Sketch the curve and describe the shape of the vase.
b. Show that $y^2 = \frac{3}{2} + 2\sin x - \frac{1}{2}\cos 2x$.
c. Find the exact volume of liquid that the vase holds.
d. Find the volume to 3 significant figures and say how many 250 ml cups it could fill.

Question 18

Question 18 – A reduction formula *(Section C · Problem solving)*

Let $I_n = \displaystyle\int_0^{\infty} x^n e^{-x}\,dx$ for $n = 0, 1, 2, \ldots$

a. Find I_0.
b. Use integration by parts to show that $I_n = nI_{n-1}$ for $n \ge 1$. Explain why the boundary term vanishes.
c. Hence find I_1, I_2, I_3, I_4.
d. State a general formula for I_n.

Question 19

Question 19 – Completing the square *(Section C · Problem solving)*

Complete the square to evaluate each integral exactly.

a. $\displaystyle\int_0^{4} \frac{dx}{\sqrt{4x - x^2}}$
b. $\displaystyle\int_2^{\infty} \frac{dx}{x^2 - 4x + 13}$
c. Find the mean value of $\dfrac{1}{x^2 - 4x + 13}$ on $[2, 5]$.

Question 20

Question 20 – Challenge: Gabriel's horn *(Section D · Challenge)*

The curve $y = \dfrac{1}{x}$ for $x \ge 1$ is rotated about the x-axis to form an infinitely long horn.

a. Show that the volume of the horn is finite and find it.
b. The curved surface area is $A = 2\pi \displaystyle\int_1^{\infty} y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$. Show that $A > 2\pi \displaystyle\int_1^{\infty} \frac{1}{x}\,dx$.

c. Deduce that the surface area is infinite.
d. Explain the paradox: you could fill the horn with π cubic units of paint but could not paint its surface.