Answers - Further calculus

Year: Further Maths

Answer to Question 1

Method: an improper integral has an infinite limit or an integrand that is unbounded at an end point. Replace the problem limit by a letter (R or a), integrate normally, then let the letter tend to its limit. If the limit is finite, the integral converges.

a. Replace ∞ by R: $\left[-\frac{1}{x}\right]_1^R$. $\int_1^R x^{-2}dx = 1 - \frac{1}{R} \to \mathbf{1}$ as $R \to \infty$. Converges. So $\int_1^\infty \frac{dx}{x^2} = 1$.

b. The integral of $\frac{1}{x}$ is $\ln x$, which has no upper bound. $\int_1^R \frac{1}{x}\,dx = \ln R \to \infty$. **Diverges.** Although $\frac{1}{x} \to 0$, it does not tend to 0 fast enough.

c. The integrand is unbounded at 0, so replace 0 by a and let $a \to 0^+$. $\int_a^1 x^{-1/2}dx = 2 - 2\sqrt{a} \to \mathbf{2}$ as $a \to 0^+$. Converges. So the area under $\frac{1}{\sqrt{x}}$ from 0 to 1 is finite and equal to 2.

d. Replace ∞ by R; $e^{-3R} \to 0$. $\int_0^R e^{-3x}dx = \frac{1}{3}\left(1 - e^{-3R}\right) \to \frac{1}{3}$. Converges. So the integral equals $\frac{1}{3}$.

Answer to Question 2

a. By parts: $\int xe^{-x}dx = -xe^{-x} - e^{-x}$. Between 0 and R: $1 - (R+1)e^{-R} \to \mathbf{1}$.

b. $\int \ln x\,dx = x\ln x - x$. From a to 1: $-1 - a\ln a + a \to -1$ since $a\ln a \to 0$.

c. The integrand is undefined at $x = 1$, inside the interval. Split: $\int_0^{1-} \frac{1}{(x-1)^2}dx = \left[\frac{-1}{x-1}\right]$ grows without limit. **Diverges** (so the answer -2 from the naive calculation is wrong).

d. $\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$. $\left[\ln\frac{x}{x+1}\right]_1^R = \ln\frac{R}{R+1} + \ln 2 \to \ln 2$.

Answer to Question 3

Method: use $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + c$ and $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a} + c$. If the quadratic is not in this form, complete the square first and then substitute $u = x + k$.

a. Here $a = 3$: $\arcsin\frac{x}{3} + C$.

b. Take out the factor 4 so that the denominator has the form $x^2 + a^2$ with $a = \frac{1}{2}$.

$\frac{1}{1+4x^2} = \frac{1}{4} \cdot \frac{1}{x^2+(1/2)^2}$. The integral is $\frac{1}{4} \cdot 2\arctan 2x = \frac{1}{2}\arctan 2x$. Between 0 and 1: $\frac{1}{2}\arctan 2 \approx 0.5536$.

c. Here $a = 2$. $\left[\arcsin \frac{x}{2}\right]_0^{\sqrt{2}} = \arcsin \frac{\sqrt{2}}{2} = \frac{\pi}{4}$.

d. $x^2 + 2x + 5 = (x+1)^2 + 4$. $\left[\frac{1}{2}\arctan \frac{x+1}{2}\right]_{-1}^{1} = \frac{1}{2}\arctan 1 - 0 = \frac{\pi}{8}$. So the integral equals $\frac{\pi}{8} \approx 0.393$.

Answer to Question 4

Method: rotating about the x-axis gives $V = \pi \int_a^b y^2\,dx$; about the y-axis, $V = \pi \int_c^d x^2\,dy$, with the limits in y. Square the function first, then integrate.

a. Square $y = \sqrt{x}$ to get $y^2 = x$. $V = \pi \int_0^4 x\,dx = \pi\left[\frac{x^2}{2}\right]_0^4 = 8\pi$ So the volume is $8\pi \approx 25.1$ cubic units.

b. Square $y = x^2$ to get $y^2 = x^4$. $V = \pi \int_0^2 x^4 dx = \pi \cdot \frac{32}{5} = \frac{32\pi}{5}$ So the volume is $\frac{32\pi}{5} \approx 20.1$ cubic units.

c. Rotating about the y-axis, write x^2 in terms of y and use the limits 0 and 4 for y. $x^2 = y$, so $V = \pi \int_0^4 y\,dy = \pi \cdot 8 = 8\pi$ So the volume is $8\pi \approx 25.1$ cubic units.

d. Square $y = e^x$ to get e^{2x}. $V = \pi \int_0^1 e^{2x} dx = \pi \cdot \frac{e^2-1}{2} = \frac{\pi(e^2-1)}{2}$ So the volume is about 10.0 cubic units.

Answer to Question 5

a. $x = x^2$ gives $x = 0$ or $x = 1$; the line $y = x$ lies above the parabola between them.

b. Rotating the line alone gives a cone of cross-section πx^2; rotating the curve alone gives a solid with cross-section $\pi(x^2)^2 = \pi x^4$. The required solid is the cone with the second solid removed, so the cross-section is a ring with area $\pi x^2 - \pi x^4$.

c. $V = \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\pi}{15}$

Answer to Question 6

Method: the mean value of f on $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$: the height of the rectangle

on $[a, b]$ with the same area as the region under the curve.

a. $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, so $\int_0^{\pi} = \frac{\pi}{2}$ and the mean is $\frac{1}{\pi} \cdot \frac{\pi}{2} = \frac{1}{2}$

b. $\int_1^3 x^2 dx = \frac{27 - 1}{3} = \frac{26}{3}$, mean $= \frac{1}{2} \cdot \frac{26}{3} = \frac{13}{3}$

c. $\int_0^2 e^{-x} dx = 1 - e^{-2}$, mean $= \frac{1 - e^{-2}}{2} \approx$ **0.4323**

d. $\int_0^1 \frac{dx}{1 + x^2} = \frac{\pi}{4}$ and $b - a = 1$, so the mean is $\frac{\pi}{4} \approx 0.785$

Answer to Question 7

a. $M(T) = \int_0^T 40e^{-0.5t} dt$.

b. $M(T) = 80\left(1 - e^{-0.5T}\right)$. At $T = 2$: $80(1 - e^{-1}) =$ **50.6 mg**.

c. The absorption carries on for ever (it never exactly stops), so we take $\lim_{T \to \infty} M(T) = \int_0^{\infty} 40e^{-0.5t} dt =$ **80 mg**, the mass of the tablet.

d. $\frac{50.57}{80} =$ **63.2%**, which is $1 - e^{-1}$.

Answer to Question 8

Method: rotating about the x-axis gives $V = \pi \int_a^b y^2 \, dx$; about the y-axis, $V = \pi \int_c^d x^2 \, dy$, with the limits in y. Square the function first, then integrate.

a. $V = \pi \int_0^h x^2 \, dy = \pi \int_0^h y \, dy = \frac{\pi h^2}{2}$.

b. At $h = 8$: $\frac{64\pi}{2} = 32\pi \approx$ **101 cm**^{3}.

c. The radius is $x = \sqrt{y} = \sqrt{6} \approx$ **2.45 cm**.

d. Volume at 6 cm: $18\pi = 56.55$ cm^3. Time $= \frac{56.55}{20} \approx$ **2.83 s**.

Answer to Question 9

Method: rotating about the x-axis gives $V = \pi \int_a^b y^2 \, dx$; about the y-axis, $V = \pi \int_c^d x^2 \, dy$, with the limits in y. Square the function first, then integrate.

a. Substitute $x = 9$ into the equation of the line. At $x = 9$, $y = 4$: radius **4 cm**, height **9 cm**.
b. Square y and integrate from 0 to 9.
$V = \pi \int_0^9 \left(\frac{4}{9}x\right)^2 dx = \pi \cdot \frac{16}{81} \cdot \frac{9^3}{3} = \pi \cdot \frac{16}{81} \cdot 243 = 48\pi$ cm^3 (≈ 150.8).
c. Use the radius and height from part a). $\frac{1}{3}\pi \cdot 16 \cdot 9 = 48\pi$. ✓ The two methods agree.
d. Mass is density times volume, with density 1.2 g per cm³. Mass $= 1.2 \times 150.8 =$ **181 g**. So the cone holds about 181 g of ice cream.

Answer to Question 10

Method: rotating about the x-axis gives $V = \pi \int_a^b y^2\,dx$; about the y-axis, $V = \pi \int_c^d x^2\,dy$, with the limits in y. Square the function first, then integrate.

a. Square both sides of $y = \sqrt{100 - x^2}$. $x^2 + y^2 = 100$ with $y \geq 0$: a semicircle with centre the origin and radius 10 m.
b. Square y and integrate from 0 to 10.
$V = \pi \int_0^{10} (100 - x^2)dx = \pi \left[100x - \frac{x^3}{3}\right]_0^{10} = \pi \left(1000 - \frac{1000}{3}\right) = \frac{2000\pi}{3} \approx 2094$ **m**^{3}.
c. $\frac{2}{3}\pi \cdot 10^3 = \frac{2000\pi}{3}$. ✓ The two methods agree.
d. Multiply the volume by the power needed per cubic metre. $0.08 \times 2094 \approx$ **168 kW**. So about 168 kW of heating is needed.

Answer to Question 11

a. $t = 0$: **14°C**. $t = 6$: $14 + 6\sin\frac{\pi}{2} =$ **20°C**. $t = 12$: $14 + 6\sin\pi =$ **14°C**.
b. $\bar{T} = \frac{1}{12}\int_0^{12} \left(14 + 6\sin\frac{\pi t}{12}\right)dt$.
c. $\int_0^{12} 6\sin\frac{\pi t}{12}dt = 6 \cdot \frac{12}{\pi} \cdot 2 = \frac{144}{\pi}$. So $\bar{T} = 14 + \frac{12}{\pi} =$ **17.8°C**.
d. The temperature is at least 14 and reaches a maximum of 20, so the mean must lie between 14 and 20; since it spends time above 14, it exceeds 14.

Answer to Question 12

Method: the mean value of f on $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$: the height of the rectangle

on $[a, b]$ with the same area as the region under the curve.

a. $\sin^2 A = \frac{1}{2}(1 - \cos 2A)$, so $P = 50(1 - \cos 240\pi t)$.

b. The period of $\cos 240\pi t$ is $\dfrac{2\pi}{240\pi} = \dfrac{1}{120}$ s, so $\frac{1}{120}$ **s** (0.00833 s).

c. $\bar{P} = 120 \displaystyle\int_0^{1/120} (50 - 50\cos 240\pi t)dt = 120\left[50t - \frac{50 \sin 240\pi t}{240\pi}\right]_0^{1/120} = 120 \cdot \frac{50}{120} =$ **50 W**.

d. Energy $= 50 \times 60 =$ **3000 J**.

Answer to Question 13

Method: an improper integral has an infinite limit or an integrand that is unbounded at an end point. Replace the problem limit by a letter (R or a), integrate normally, then let the letter tend to its limit. If the limit is finite, the integral converges.

a. Integrate the current from 0 to 1. $Q = \displaystyle\int_0^1 3e^{-2t}dt = \frac{3}{2}\left(1 - e^{-2}\right) \approx$ **1.30 C**.

b. Let the upper limit tend to infinity: $e^{-2R} \to 0$. $\displaystyle\int_0^\infty 3e^{-2t}dt =$ **1.5 C**. So in total 1.5 coulombs flow.

c. Set the charge equal to half of 1.5 C and solve with logarithms. $\frac{3}{2}\left(1 - e^{-2T}\right) = 0.75$, $e^{-2T} = \frac{1}{2}$, $T = \frac{1}{2}\ln 2 \approx$ **0.347 s**. So half the charge has flowed after about 0.35 s.

d. Use the mean value formula on $[0, 1]$. Mean $= \dfrac{1.297}{1} =$ **1.30 A** (the mean over one second equals the charge in that second).

Answer to Question 14

Method: an improper integral has an infinite limit or an integrand that is unbounded at an end point. Replace the problem limit by a letter (R or a), integrate normally, then let the letter tend to its limit. If the limit is finite, the integral converges.

a. $\dfrac{1}{\pi}[\arctan x]_{-\infty}^{\infty} = \dfrac{1}{\pi}\left(\dfrac{\pi}{2} + \dfrac{\pi}{2}\right) =$ **1**.

b. $\dfrac{1}{\pi}[\arctan x]_0^1 = \dfrac{1}{\pi} \cdot \dfrac{\pi}{4} = \dfrac{1}{4}$.

c. $\dfrac{1}{\pi}\left(\dfrac{\pi}{2} - \dfrac{\pi}{4}\right) = \dfrac{1}{4}$.

d. $\int_0^R \dfrac{x}{\pi(1 + x^2)}dx = \dfrac{1}{2\pi}\ln(1 + R^2) \to \infty$, so the integral on each side diverges. The mean is undefined even though the distribution is a valid one.

Answer to Question 15

a. For $p \neq 1$, $\int_1^R x^{-p}dx = \dfrac{R^{1-p}-1}{1-p}$. If $p > 1$ then $R^{1-p} \to 0$ and the limit is $\dfrac{1}{p-1}$. If p then $R^{1-p} \to \infty$. If $p = 1$ the integral is $\ln R \to \infty$. **Converges only for $p > 1$.**

b. $\dfrac{1}{p-1} = \dfrac{1}{2}$, so $p = 3$.

c. $\int_a^1 x^{-p}dx = \dfrac{1-a^{1-p}}{1-p}$ for $p \neq 1$. If p then $a^{1-p} \to 0$ and the limit is $\dfrac{1}{1-p}$. If $p \geq 1$ the limit does not exist (for $p = 1$ it is $-\ln a \to \infty$). **Converges only for p.**

d. $x = u^2$, $dx = 2u\,du$, $\sqrt{x} = u$: $\displaystyle\int_0^\infty \frac{2u\,du}{u(1+u^2)} = 2[\arctan u]_0^\infty = \pi$.

Answer to Question 16

Method: rotating about the x-axis gives $V = \pi \displaystyle\int_a^b y^2\,dx$; about the y-axis, $V = \pi \displaystyle\int_c^d x^2\,dy$, with the limits in y. Square the function first, then integrate. For the region between two curves, use $\pi \displaystyle\int (y_1^2 - y_2^2)\,dx$, where y_1 is the outer curve.

a. Solve the two equations simultaneously. $x^2 = 2x$ gives $x = 0$ or 2: $(0,0)$ **and** $(2,4)$.

b. The line is above the curve between the two points, so integrate line minus curve. $\displaystyle\int_0^2 (2x - x^2)dx = \left[x^2 - \frac{x^3}{3}\right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$. So the area of R is $\frac{4}{3}$.

c. The outer radius is $2x$ and the inner radius is x^2. Do not square the difference. $V = \pi \displaystyle\int_0^2 \left((2x)^2 - (x^2)^2\right)dx = \pi\left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^2 = \pi\left(\frac{32}{3} - \frac{32}{5}\right) = \frac{64\pi}{15}$. So the volume is $\frac{64\pi}{15} \approx 13.4$.

d. 1 dm³ is 1 litre. $\dfrac{64\pi}{15} \approx$ **13.4 litres**.

Answer to Question 17

a. The curve starts at $y = 1$, rises to $y = 2$ at $x = \frac{\pi}{2}$ and returns to $y = 1$ at $x = \pi$. The vase is **widest in the middle** (radius 2 cm) and has radius 1 cm at both ends, like a barrel.

b. $y^2 = 1 + 2\sin x + \sin^2 x = 1 + 2\sin x + \frac{1}{2}(1 - \cos 2x) = \frac{3}{2} + 2\sin x - \frac{1}{2}\cos 2x$.

c. $V = \pi\left[\frac{3}{2}x - 2\cos x - \frac{1}{4}\sin 2x\right]_0^\pi = \pi\left(\frac{3\pi}{2} + 2 - (-2)\right) = \pi\left(4 + \frac{3\pi}{2}\right)$.

d. $V = \pi(4 + 4.712) \approx$ **27.4 cm^3**. A cup holds 250 cm^3, so the vase fills **0.11 of a**

cup, far less than one cup: the vase is small.

Answer to Question 18

a. $I_0 = \int_0^\infty e^{-x}dx = [-e^{-x}]_0^\infty = \mathbf{1}$.

b. Let $u = x^n$, $dv = e^{-x}dx$: $I_n = [-x^n e^{-x}]_0^\infty + n\int_0^\infty x^{n-1}e^{-x}dx$. The boundary term is 0 at $x = 0$ (for $n \geq 1$) and tends to 0 as $x \to \infty$ because exponentials grow faster than any power. Hence $I_n = nI_{n-1}$.

c. $I_1 = 1 \cdot 1 = \mathbf{1}$, $I_2 = 2 \cdot 1 = \mathbf{2}$, $I_3 = 3 \cdot 2 = \mathbf{6}$, $I_4 = 4 \cdot 6 = \mathbf{24}$.

d. $I_n = n!$. (This is the idea behind the gamma function.)

Answer to Question 19

Method: use $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} + c$ and $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\arctan\frac{x}{a} + c$. If the quadratic is not in this form, complete the square first and then substitute $u = x + k$.

a. $4x - x^2 = 4 - (x-2)^2$, so the integral is $\left[\arcsin\frac{x-2}{2}\right]_0^4 = \arcsin 1 - \arcsin(-1) = \frac{\pi}{2} + \frac{\pi}{2} = \pi$. (It is improper because the integrand is unbounded at both ends, yet converges.)

b. $x^2 - 4x + 13 = (x-2)^2 + 9$. $\left[\frac{1}{3}\arctan\frac{x-2}{3}\right]_2^\infty = \frac{1}{3} \cdot \frac{\pi}{2} = \frac{\pi}{6}$.

c. $\int_2^5 \frac{dx}{(x-2)^2 + 9} = \frac{1}{3}\arctan 1 = \frac{\pi}{12}$. Interval length 3. Mean $= \frac{\pi}{36} \approx 0.0873$.

Answer to Question 20

a. $V = \pi\int_1^\infty \frac{1}{x^2}dx = \pi \cdot 1 = \pi$.

b. $\frac{dy}{dx} = -\frac{1}{x^2}$, so $\sqrt{1 + \frac{1}{x^4}} > 1$ and $y\sqrt{\dots} > \frac{1}{x}$ for $x \geq 1$. Hence $A > 2\pi\int_1^\infty \frac{1}{x}\,dx$.

c. $\int_1^\infty \frac{1}{x}\,dx$ diverges (it is $\ln R \to \infty$), so by comparison A **diverges**.

d. A finite volume of paint spreads over a surface of infinite area if the layer of paint gets thinner and thinner. A real layer cannot become thinner than one molecule, so the paradox disappears in the physical world; it is a statement about idealised mathematics only.