Test answers - Differential equations

Year: Further Maths · Duration: 60 minutes · /20

Answer to Question 1 (4 marks)

Method (integrating factor): write the equation as $\frac{dy}{dx} + Py = Q$, multiply by $I = e^{\int P\,dx}$; the left-hand side becomes $\frac{d}{dx}(Iy)$. Integrate both sides, remember the constant, then divide by I. Mark scheme as shown in brackets.

a. $I = e^{-3\ln x} = x^{-3}$ [1]. $\frac{d}{dx}(yx^{-3}) = x^{-1}$ [1]. $yx^{-3} = \ln x + C$ [1]. $y(1) = 2$ gives $C = 2$: $y = x^3(\ln x + 2)$ [1]

Check: $y' = 3x^2(\ln x + 2) + x^2$, and $y' - \frac{3y}{x} = x^2$, as required.

The integrating factor uses $P = -\frac{3}{x}$, not $+\frac{3}{x}$: the sign matters.

Answer to Question 2 (6 marks)

Method: the general solution of $ay'' + by' + cy = f(x)$ is the complementary function (solution with $f = 0$) plus a particular integral, whose form copies $f(x)$. Use the initial conditions only on the complete general solution. Mark scheme as shown in brackets.

a. CF: $m^2 - 5m + 6 = 0$, $m = 2, 3$: $Ae^{2x} + Be^{3x}$ [1]. PI: $y = ke^x$ gives $k(1 - 5 + 6) = 4$ [1], $k = 2$ [1]. General solution $y = Ae^{2x} + Be^{3x} + 2e^x$ [1]

b. $A + B + 2 = 5$ and $2A + 3B + 2 = 7$ [1] give $A + B = 3$, $2A + 3B = 5$, $B = -1$, $A = 4$: $y = 4e^{2x} - e^{3x} + 2e^x$ [1]

Answer to Question 3 (4 marks)

Method: for $ay'' + by' + cy = 0$, solve the auxiliary equation $am^2 + bm + c = 0$. Two real roots give $Ae^{m_1 x} + Be^{m_2 x}$; a repeated root gives $(A + Bx)e^{mx}$; complex roots $p \pm qi$ give $e^{px}(A\cos qx + B\sin qx)$. Mark scheme as shown in brackets.

a. $m^2 + 4m + 20 = 0$, $m = -2 \pm 4i$ [1]. $y = e^{-2x}(A\cos 4x + B\sin 4x)$ [1]. $y(0) = A = 2$ [1]. $y'(0) = -2A + 4B = 0$, $B = 1$: $y = e^{-2x}(2\cos 4x + \sin 4x)$ [1]

The motion is a damped oscillation: the factor e^{-2x} makes the amplitude shrink quickly.

The roots come from $m = \frac{-4 \pm \sqrt{16 - 80}}{2} = \frac{-4 \pm 8i}{2}$.

Answer to Question 4 (3 marks)

Method: $\ddot{x} = -\omega^2 x$ is simple harmonic motion with period $\dfrac{2\pi}{\omega}$, maximum speed $a\omega$ and $v^2 = \omega^2(a^2 - x^2)$. Here $\omega^2 = 25$ and $a = 0.2$. Mark scheme as shown in brackets.

a. $\omega = 5$, period $\dfrac{2\pi}{5} =$ **1.26 s** [1]
b. $v_{max} = \omega a = 5 \times 0.2 =$ **1 m s^{-1}** [1]
c. $v^2 = 25(0.04 - 0.0144) = 0.64$, $v =$ **0.8 m s^{-1}** [1]

Check (c): at $x = 0.12$, $\sqrt{a^2 - x^2} = \sqrt{0.0256} = 0.16$, so $v = 5 \times 0.16 = 0.8$ m s^{-1} ✓. The period does not depend on the amplitude, a key property of simple harmonic motion. The exact period $\frac{2\pi}{5}$ s is also accepted.

Answer to Question 5 (3 marks)

Method: substitute $\theta = T - 15$, the excess temperature over the garage, so that $\dfrac{d\theta}{dt} = -0.05\theta$, an exponential decay. Use $T = 75$ at $t = 0$ to find the constant. Mark scheme as shown in brackets.

a. $\dfrac{d}{dt}(T - 15) = -0.05(T - 15)$ gives $T - 15 = Ke^{-0.05t}$ [1]; $K = 60$ [1]
b. $30 = 15 + 60e^{-0.05t}$, $e^{-0.05t} = \frac{1}{4}$, $t = 20\ln 4 =$ **27.7 min** [1]

So the engine cools to 30°C after about 28 minutes; the excess temperature halves every $20\ln 2 \approx 13.9$ minutes.

Separating variables, $\ln|T - 15| = -0.05t + c$, gives the same result.