Lesson - Differential equations

Year: Further Maths

Differential equations describe how a quantity changes. At A level Further Maths you solve first-order linear equations with an integrating factor, second-order linear equations with constant coefficients, and use them to model motion, cooling and mixing.

Starter activity

A bath is filling from a tap at a steady rate, but the plug is not quite in, so water also drains away at a rate that depends on how deep it is. At first the level rises quickly, then more slowly, and eventually it stops rising. Why does it stop? Sketch a graph of depth against time and mark the final depth. Write a sentence in words that describes how the rate of change of the depth depends on the depth. You have just described a differential equation, and in this chapter you learn to solve this kind of equation exactly.

First-order linear equations

Definition: integrating factor

For $\frac{dy}{dx} + P(x)y = Q(x)$, the **integrating factor** is $I(x) = e^{\int P\,dx}$. Multiplying through by I makes the left side $\frac{d}{dx}(Iy)$.

Method: integrating factor

1. Divide so that the coefficient of $\frac{dy}{dx}$ is 1, then read off $P(x)$.
2. Find $I = e^{\int P\,dx}$ (no constant needed).
3. Write $\frac{d}{dx}(Iy) = IQ$ and integrate, adding $+c$.
4. Divide by I to make y the subject, then use any initial condition.

Example 1

Solve $x\frac{dy}{dx} + 2y = \sin x$ for $x > 0$.

Divide by x: $\frac{dy}{dx} + \frac{2}{x}y = \frac{\sin x}{x}$. $I = e^{2\ln x} = x^2$. So $(x^2y)' = x\sin x$, giving $x^2y = \sin x - x\cos x + c$ and $y = \frac{\sin x - x\cos x + c}{x^2}$.

Second-order linear equations with constant coefficients

Rule: general solution

For $a\dfrac{d^2y}{dx^2}+b\dfrac{dy}{dx}+cy=f(x)$, the general solution is $y=$ **complementary function** (CF) $+$ **particular integral** (PI). The CF solves the homogeneous equation ($f=0$). Try $y=e^{mx}$ to get the **auxiliary equation** $am^2+bm+c=0$.

Key fact: the complementary function

Roots of auxiliary equation	Complementary function
Real, distinct m_1, m_2	$Ae^{m_1x}+Be^{m_2x}$
Repeated m	$(A+Bx)e^{mx}$
Complex $p\pm qi$	$e^{px}(A\cos qx+B\sin qx)$

Key fact: trial particular integrals

$f(x)$	Try PI
constant k	λ
$kx+l$	$\lambda x+\mu$
ke^{px}	λe^{px}
$k\cos px$ or $k\sin px$	$\lambda\cos px+\mu\sin px$

If the trial is already part of the CF, multiply it by x (or x^2 for a repeated root).

Method: solving a second-order equation

1. Write and solve the auxiliary equation to get the CF.
2. Choose a trial PI from the table, differentiate it twice and substitute.
3. Compare coefficients to find λ, μ.
4. State $y=\mathrm{CF}+\mathrm{PI}$.
5. Only now use the initial or boundary conditions to find A and B.

Example 2

Find the general solution of $y''-3y'+2y=4x$.

CF: $m^2-3m+2=0\Rightarrow m=1,2$, so Ae^x+Be^{2x}. PI: try $y=px+q$:
$-3p+2px+2q=4x\Rightarrow p=2,\ q=3$. General solution $y=Ae^x+Be^{2x}+2x+3$.

Example 3: resonance

Solve $y'' - 4y = 8e^{2x}$.

CF: $Ae^{2x} + Be^{-2x}$. Since e^{2x} is in the CF, try $y = \lambda x e^{2x}$: $y' = \lambda(1 + 2x)e^{2x}$, $y'' = \lambda(4 + 4x)e^{2x}$, so $y'' - 4y = 4\lambda e^{2x} = 8e^{2x} \Rightarrow \lambda = 2$. General solution $y = Ae^{2x} + Be^{-2x} + 2xe^{2x}$.

Watch out

- Do not apply the boundary conditions to the CF before adding the PI.
- A PI trial for $\cos px$ needs both $\cos$ and $\sin$.
- Forgetting to multiply the trial by x when it clashes with the CF.
- Not dividing so that the coefficient of $\frac{dy}{dx}$ is 1 before finding I.

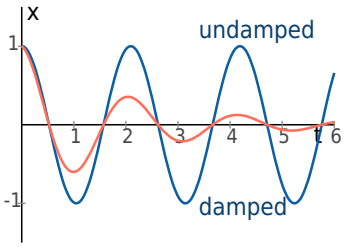
Simple harmonic motion and modelling

Definition: simple harmonic motion

$\frac{d^2x}{dt^2} = -\omega^2 x$ is the equation of **simple harmonic motion**. Its solution is $x = A\cos\omega t + B\sin\omega t = R\cos(\omega t - \varepsilon)$, with amplitude R, period $\frac{2\pi}{\omega}$ and maximum speed $R\omega$.

Key fact: damping

A mass m on a spring of stiffness k gives $m\ddot{x} = -kx$, so $\omega = \sqrt{k/m}$. With damping, $\ddot{x} + 2\zeta\dot{x} + \omega^2 x = 0$ and the auxiliary roots decide the motion: real distinct is **overdamped**, repeated is **critically damped**, complex is **underdamped** (a decaying oscillation).



Blue: undamped simple harmonic motion x = cos 3t. Coral: lightly damped oscillation, amplitude decaying like e^(-t/2).

Example 4

Describe the motion $\ddot{x} + 2\dot{x} + 5x = 0$.

The roots are $-1 \pm 2i$, so $x = e^{-t}(A\cos 2t + B\sin 2t)$: an oscillation of angular

frequency 2 whose amplitude decays like e^{-t} (underdamped). Critical damping would need a repeated root, as in $\ddot{x} + 2\dot{x} + x = 0$ with $x = (A + Bt)e^{-t}$.

Note: other models

Newton's law of cooling is $\frac{dT}{dt} = -k(T - T_0)$, with solution $T = T_0 + (T_1 - T_0)e^{-kt}$. Mixing problems use $\frac{dm}{dt} = \text{rate in} - \text{rate out}$, where the rate out depends on the current amount. State the final answer in the context of the problem, with units.

Worked example 5: a first order equation with a trigonometric forcing term

Solve $\dfrac{dy}{dx} + 2y = \sin x$.

The integrating factor is e^{2x}, so $\dfrac{d}{dx}\left(ye^{2x}\right) = e^{2x}\sin x$. To integrate the right-hand side, use integration by parts twice, or recognise that $\int e^{2x}\sin x\,dx = \frac{1}{5}e^{2x}(2\sin x - \cos x)$. Then $ye^{2x} = \frac{1}{5}e^{2x}(2\sin x - \cos x) + C$, and dividing by e^{2x} gives $y = \frac{1}{5}(2\sin x - \cos x) + Ce^{-2x}$. The term with C dies away as x increases, and what remains is a steady oscillation, the response of the system to the sine forcing.

Note: two kinds of solution

A differential equation of order n has a general solution that contains n arbitrary constants. Each choice of the constants gives a particular solution, and a set of conditions on y at a single point, such as the value of y and the value of y' at $x = 0$, picks out exactly one of them. It is a common mistake to substitute the initial conditions into the complementary function alone. The conditions apply to the whole of the general solution, which is the complementary function plus the particular integral, so you must find the particular integral first and only then use the conditions.

Method: second order equations with constant coefficients

1. Write down the auxiliary equation $am^2 + bm + c = 0$ and solve it to find the complementary function.
2. Choose the form of the particular integral from the right-hand side: a constant for a constant, a polynomial of the same degree for a polynomial, ke^{px} for an exponential, and $a\cos qx + b\sin qx$ for a sine or cosine.
3. If the trial form already appears in the complementary function, multiply it by x (and by x^2 if it appears twice).
4. Substitute the trial form into the equation and compare coefficients to find the constants.
5. Write the general solution as the sum of the two parts and then apply the initial conditions.

Worked example 6: a particular integral that contains a product

Find the general solution of $y'' - 2y' + y = e^x$.

The auxiliary equation is $(m-1)^2 = 0$, with the repeated root 1, so the complementary function is $(A + Bx)e^x$. Both e^x and xe^x appear in it, so the trial form must be $y = kx^2e^x$. Differentiating, $y' = k(x^2 + 2x)e^x$ and $y'' = k(x^2 + 4x + 2)e^x$. Substituting gives $k\left[(x^2 + 4x + 2) - 2(x^2 + 2x) + x^2\right]e^x = 2ke^x$. Hence $2k = 1$, $k = \frac{1}{2}$, and the general solution is $y = \left(A + Bx + \frac{1}{2}x^2\right)e^x$.

Key fact: the three types of damping

For the damped oscillator $\ddot{x} + 2\lambda\dot{x} + \omega^2 x = 0$ the auxiliary equation has discriminant $4\lambda^2 - 4\omega^2$. If λ the roots are complex, the motion is an oscillation whose amplitude decays like $e^{-\lambda t}$, and the damping is light (underdamped). If $\lambda = \omega$ the root is repeated, the solution is $(A + Bt)e^{-\lambda t}$, and the system returns to equilibrium in the shortest time without oscillating: this is critical damping. If $\lambda > \omega$ the roots are real, distinct and negative, and the system creeps back slowly, which is heavy damping (overdamped). Door closers and car suspensions are designed to be close to critical damping.

Worked example 7: setting up a model

A population of bacteria grows at a rate proportional to its size, but 300 bacteria per hour are removed by a filter. Write a differential equation and solve it if there are 2000 bacteria at the start and the growth rate is 10% per hour.

Let N be the number of bacteria after t hours. The growth contributes $0.1N$ and the filter removes 300, so $\dfrac{dN}{dt} = 0.1N - 300$. The equilibrium is at $N = 3000$. Writing the equation as $\dfrac{dN}{dt} - 0.1N = -300$ and using the integrating factor $e^{-0.1t}$ gives $N = 3000 + Ke^{0.1t}$. Since $N(0) = 2000$, $K = -1000$, so $N = 3000 - 1000e^{0.1t}$. This reaches zero when $e^{0.1t} = 3$, that is after $10\ln 3 \approx 11$ hours: the filter wins because the starting population is below the equilibrium value.

Watch out: frequent mistakes

- Forgetting to divide the equation by the coefficient of $\dfrac{dy}{dx}$ before finding the integrating factor. The equation must be in the standard form $\dfrac{dy}{dx} + Py = Q$.
- Dropping the constant of integration when integrating $Ie^{\cdots}$, or adding it on the wrong side so it is not multiplied by the integrating factor correctly.
- Using the initial conditions on the complementary function alone.
- Forgetting to multiply the trial function by x when it clashes with the

complementary function.
- Writing $e^{\sqrt{-1}\,x}$ solutions with complex numbers when the question asks for a real answer: use $\cos$ and $\sin$ instead.

Note: how to check a solution

You can always check an answer to a differential equation by differentiating it and substituting back into the original equation. For a first order equation this takes only a few lines and is worth doing in an examination if you have spare time. For a second order equation, check that your solution satisfies both initial conditions as well. A solution that does not satisfy the equation is wrong, and the check will almost always reveal where a sign has gone astray.

Applications of differential equations

Almost every law of physics is a differential equation. Newton's second law relates the acceleration of a body to the forces on it, and when the force depends on position or velocity the law becomes a differential equation for the position. Radioactive decay, the cooling of coffee, the charging of a phone battery and the spread of a medicine through the body are all described by first order equations. Pendulums, springs, vibrating bridges, and electrical circuits with coils and capacitors lead to second order equations of the type studied here. Learning to set up such equations from a written description is a very valuable skill, because the algebra of solving them is the easy part once the model is correct.

Method: modelling with differential equations

1. Define the variables and units clearly, for example T for temperature in degrees Celsius and t for time in minutes.
2. Translate the sentence about a rate of change into an equation: "rate of change of T is proportional to $T - 20$" becomes $\dfrac{dT}{dt} = -k(T - 20)$.
3. Choose the correct sign by thinking about whether the quantity should increase or decrease.
4. Solve the equation, use the initial conditions to find the constants, and use any other information to find the parameter such as k.
5. Answer the question in context, with units, and comment on whether the model is sensible for large values of t.

Why the integrating factor works

The integrating factor can feel like a trick, so it helps to see the idea behind it. The left-hand side of the equation $\dfrac{dy}{dx} + Py$ looks almost like the result of the product rule, but not quite. If we multiply the whole equation by a suitable function $I(x)$, then we

want $I\dfrac{dy}{dx} + IPy$ to be exactly the derivative of the product Iy. The product rule says that the derivative of Iy is $I\dfrac{dy}{dx} + \dfrac{dI}{dx}y$, so we need $\dfrac{dI}{dx} = PI$. That is a simple equation whose solution is $I = e^{\int P\,dx}$. Once the left-hand side is a perfect derivative, we integrate both sides and the problem is solved. You do not need to include a constant of integration when you find I, because any multiple of an integrating factor is also an integrating factor, and the extra constant would cancel at the end.

Worked example 8: forced oscillation with a damper

A mass on a spring satisfies $\ddot{x} + 2\dot{x} + 5x = 10\cos t$. Find the steady state motion.

The complementary function is $e^{-t}(A\cos 2t + B\sin 2t)$ and it decays to zero, so after a short time only the particular integral remains. Try $x = a\cos t + b\sin t$. Then $\dot{x} = -a\sin t + b\cos t$ and $\ddot{x} = -a\cos t - b\sin t$. Substituting, the coefficient of $\cos t$ is $-a + 2b + 5a = 4a + 2b$, which must equal 10, and the coefficient of $\sin t$ is $-b - 2a + 5b = 4b - 2a$, which must equal 0. So $a = 2b$ and $10b = 10$, giving $b = 1$ and $a = 2$. The steady state motion is $x = 2\cos t + \sin t$, an oscillation of amplitude $\sqrt{5}$ at the same frequency as the forcing. The transient, which is the part that dies away, depends on the starting conditions, but the steady state does not.

Reading a model critically

When you have solved a modelling question, always ask whether the answer makes sense. A cooling object cannot become colder than its surroundings, so the temperature must level off at room temperature. A drug concentration cannot become negative. A mass on a spring with damping should lose energy, so its oscillations should die away rather than grow. In an examination you may be asked to comment on the limitations of the model: for example, Newton's law of cooling assumes that the surroundings stay at a constant temperature, and the model for a skydiver assumes that air resistance is proportional to speed, which is not accurate at high speeds. Writing one sensible sentence about this can earn an extra mark.

Method: a checklist before you finish

1. Have you included the constants of integration, and found them using every condition given?
2. Does your solution satisfy the original differential equation when you substitute a simple value such as $x = 0$?
3. Have you given the answer in the form that was asked for, such as $y = \ldots$, and with units if the question is a model?
4. For long-term behaviour, have you looked at what happens to each term as the variable becomes very large?

Comparing the two types of equation

First order and second order equations look different, but they share a common structure. In both cases the general solution is the sum of a part that solves the simpler homogeneous equation, which has zero on the right-hand side, and a particular solution that responds to the forcing term. The homogeneous part usually contains the constants of integration and describes how the system would behave if left alone, for example the natural decay of a temperature or the natural swing of a spring. The particular part describes how the system is pushed by an outside influence. In many practical problems the homogeneous part decays quickly and is called the transient, leaving the particular part as the long-term or steady state behaviour. Recognising this pattern helps you to predict the shape of a solution before you calculate it.

Worked example 9: a second order model for a pendulum

A pendulum swinging through small angles satisfies $\ddot{\theta} = -\dfrac{g}{L}\theta$. A clockmaker wants a pendulum with a period of exactly 2 seconds. What length is needed? Take $g = 9.8$ m s^{-2}.

The equation is simple harmonic with $\omega^2 = \dfrac{g}{L}$, so the period is $2\pi\sqrt{\dfrac{L}{g}}$. Setting this equal to 2 gives $\sqrt{\dfrac{L}{g}} = \dfrac{1}{\pi}$, so $L = \dfrac{g}{\pi^2} = \dfrac{9.8}{9.8696} \approx 0.993$ metres. This is why a clock with a one-second beat uses a pendulum of almost exactly one metre. The model assumes that the angle stays small so that $\sin\theta \approx \theta$, and it ignores air resistance. For a swing of 20 degrees the error in the period is only about one percent, so the approximation is excellent for clocks.

Watch out: signs in the auxiliary equation

When the equation is written with all terms on the left, such as $y'' - 3y' - 4y = 0$, the auxiliary equation is $m^2 - 3m - 4 = 0$ with the same signs. Students sometimes change the sign of the middle term by mistake or forget that the constant term is the coefficient of y, not of y''. It is also easy to confuse the roots of the auxiliary equation with the solution itself. If the roots are $m = 4$ and $m = -1$, the solution is $Ae^{4x} + Be^{-x}$, and the numbers 4 and -1 appear in the exponents. If the roots are $-2 \pm 3i$, the real part -2 appears in the exponential and the imaginary part 3 appears inside the cosine and sine.

Did you know?

Differential equations were invented by Isaac Newton and Gottfried Leibniz in the seventeenth century, when they developed calculus. Newton used them to show that a single law of gravity explains both the fall of an apple and the orbit of the Moon. Today, the same equations control the design of aircraft, the forecasting of the

weather, and the models used to track the spread of diseases. When the pendulum clock was invented by Christiaan Huygens in 1656, its small oscillations were the first example of simple harmonic motion applied to measuring time accurately.

Otto's tip

Always check your final solution by substituting it back into the equation and the initial conditions.