Worksheet - Differential equations

Year: Further Maths · 20

Show your working.

Question 1

Question 1 – First order equations and the integrating factor *(Section A · Skills practice)*

Solve each differential equation by finding an integrating factor $I = e^{\int P\,dx}$, where the equation is written as $\frac{dy}{dx} + Py = Q$.

a. $\frac{dy}{dx} + 3y = 6$

b. $x\frac{dy}{dx} + 2y = 4x$ for $x > 0$

c. $\frac{dy}{dx} + y\tan x = \cos x$ for $-\frac{\pi}{2}$

d. $\frac{dy}{dx} + 2y = e^{-x}$, given that $y = 3$ when $x = 0$

Question 2

Question 2 – The auxiliary equation *(Section A · Skills practice)*

Find the general solution of each equation by writing down and solving the auxiliary equation.

a. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

b. $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$

c. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$

d. $\frac{d^2y}{dx^2} + 16y = 0$

Question 3

Question 3 – Particular integrals *(Section A · Skills practice)*

In each part find a particular integral, then write down the general solution.

a. $y'' - 4y' + 3y = 6x$

b. $y'' + y' - 6y = 10e^{2x}$ (check the roots of the auxiliary equation first)
c. $y'' + 9y = \sin 2x$

Question 4

Question 4 – Initial conditions *(Section A · Skills practice)*

Solve $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$ given that $y = 1$ and $\frac{dy}{dx} = 5$ when $x = 0$.

a. Write down the auxiliary equation and solve it.
b. Write down the general solution.
c. Use the two conditions to find A and B.
d. State the value of y when $x = \ln 2$.

Question 5

Question 5 – Simple harmonic motion *(Section A · Skills practice)*

A particle moves so that $\frac{d^2x}{dt^2} = -16x$, with $x = 3$ and $\frac{dx}{dt} = 0$ when $t = 0$ (metres and seconds).

a. Write down the general solution and find the particular solution.
b. State the amplitude and the period of the motion.
c. Find the greatest speed of the particle.
d. Find the speed of the particle when $x = 2$.

Question 6

Question 6 – Checking and finding the general solution *(Section A · Skills practice)*

Consider the equation $y'' + 4y' + 4y = 0$.

a. Show that $y = e^{-2x}$ satisfies the equation.
b. Show that $y = xe^{-2x}$ satisfies the equation.
c. Write down the general solution and explain why both functions are needed.
d. Find the solution with $y(0) = 1$ and $y'(0) = 0$.

Question 7

Question 7 – Cooling a cup of tea *(Section B · Real-life practice)*

A cup of tea at 90°C is left in a room at a constant 20°C. Newton's law of cooling says that the temperature T degrees Celsius after t minutes satisfies $\frac{dT}{dt} = -k(T - 20)$ for a constant $k > 0$. After 5 minutes the tea is at 60°C.

a. Solve the differential equation with $T(0) = 90$.
b. Show that $k = \frac{1}{5}\ln\frac{7}{4} \approx 0.112$.
c. How long after being made does the tea reach 40°C, the temperature at which Jo likes to drink it?
d. What happens to T as $t \to \infty$?

Question 8

Question 8 – Medicine in the bloodstream *(Section B · Real-life practice)*

A drug is fed into a patient's blood by a drip at a constant rate. The concentration C mg per litre at time t hours satisfies $\frac{dC}{dt} = 4 - 0.2C$, with $C = 0$ at $t = 0$.

a. Write the equation in the form $\frac{dC}{dt} + PC = Q$ and find the integrating factor.
b. Solve the equation to show that $C = 20\left(1 - e^{-0.2t}\right)$.
c. Find the limiting concentration as $t \to \infty$.
d. The doctor wants a concentration of 15 mg per litre. After how many hours does it reach this value?

Question 9

Question 9 – The salt tank *(Section B · Real-life practice)*

A tank holds 200 litres of water in which 10 kg of salt is dissolved. Brine containing 0.1 kg of salt per litre flows in at 4 litres per minute. The well-stirred mixture flows out at 4 litres per minute, so the volume stays at 200 litres.

a. Show that the mass Q kg of salt in the tank after t minutes satisfies $\frac{dQ}{dt} = 0.4 - \frac{Q}{50}$.
b. Solve the equation to show that $Q = 20 - 10e^{-t/50}$.
c. How long does it take for the tank to hold 15 kg of salt?
d. Find the limit of Q and explain it in terms of the concentration.

Question 10

Question 10 – Charging a capacitor *(Section B · Real-life practice)*

In a circuit a capacitor of $500\ \mu$F is charged through a resistor of $2000\ \Omega$ from a 12 V battery. The voltage V volts across the capacitor satisfies $RC\frac{dV}{dt} + V = 12$, with $V = 0$ when $t = 0$ seconds.

a. Calculate RC and write the equation in standard form.
b. Solve for V in terms of t.
c. How long does the capacitor take to reach 9 V?

d. Find $\frac{dV}{dt}$ at $t = 0$ and describe what it tells you about the start of charging.

Question 11

Question 11 – A scooter suspension *(Section B · Real-life practice)*

The suspension of a scooter is modelled by a mass of 2 kg attached to a spring and a damper. The displacement x metres from equilibrium after t seconds satisfies $2\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 26x = 0$. The scooter is released at $x = 0.5$ from rest.

a. Show that the auxiliary equation has roots $-2 \pm 3i$ and say whether the damping is light, critical or heavy.
b. Find x in terms of t.
c. State the period of the oscillations and the factor by which the amplitude is reduced after each period.
d. Explain what would change if the damper were replaced by a stronger one with $c = 20$ in place of 8.

Question 12

Question 12 – Savings and withdrawals *(Section B · Real-life practice)*

Ben has £40 000 in an account that earns 4% interest per year, added continuously, and he withdraws £2000 per year at a steady rate. The balance £A after t years satisfies $\frac{dA}{dt} = 0.04A - 2000$.

a. Find the equilibrium balance, at which the balance would neither grow nor fall.
b. Solve the equation with $A(0) = 40000$.
c. After how many years is the account empty?
d. What initial balance would keep the account steady for ever?

Question 13

Question 13 – Marching over a footbridge *(Section B · Real-life practice)*

A footbridge deck behaves like a mass on a spring and its vertical displacement x metres satisfies $\frac{d^2x}{dt^2} + 4x = F(t)$, where F is the forcing due to the footsteps of walkers.

a. Find the complementary function.
b. The walkers produce $F = 10\cos 3t$. Find a particular integral of the form $a\cos 3t$ and write down the general solution.
c. If the walkers instead produce $F = 8\cos 2t$, explain why the form $a\cos 2t + b\sin 2t$ fails, and show that $x = 2t\sin 2t$ is a particular integral.

d. What does the second answer say about the displacement as time passes, and why do soldiers break step when they cross a bridge?

Question 14

Question 14 – The skydiver *(Section B · Real-life practice)*

A skydiver of mass 70 kg falls from rest. The air resistance is proportional to speed and the speed v m s^{-1} after t seconds satisfies $\dfrac{dv}{dt} = 9.8 - 0.2v$.

a. Find the terminal speed.
b. Solve the equation to show that $v = 49\left(1 - e^{-0.2t}\right)$.
c. Find the speed after 10 seconds.
d. Integrate to find the distance fallen after 10 seconds.

Question 15

Question 15 – A curve through a point *(Section C · Problem solving)*

A curve satisfies the differential equation $x\dfrac{dy}{dx} - 2y = x^4$ for $x > 0$.

a. Write the equation in the form $\dfrac{dy}{dx} + Py = Q$ and find the integrating factor.
b. Find the general solution.
c. The curve passes through $(1, 3)$. Find its equation.
d. Find the value of y at $x = 2$ and show that the curve has no stationary point for $x > 0$.

Question 16

Question 16 – Complementary function plus particular integral *(Section C · Problem solving)*

Consider $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 5y = 5x + 12$.

a. Find the complementary function.
b. Find a particular integral of the form $ax + b$.
c. Write down the general solution.
d. Find the solution with $y = 3$ and $\dfrac{dy}{dx} = 0$ at $x = 0$, and describe the behaviour of y for large x.

Question 17

Question 17 – Choosing the damping *(Section C · Problem solving)*

A mass of 1 kg moves on a spring of stiffness 25 N m^{-1} and is damped by a force $c\dot{x}$ newtons. Its displacement x metres satisfies $\ddot{x} + c\dot{x} + 25x = 0$.

a. Find the value of c for which the damping is critical.
b. For $c = 6$ find the roots of the auxiliary equation and say whether the motion oscillates.
c. For $c = 12$ show that there are two distinct real roots and write the general solution.
d. For $c = 10$, with $x(0) = 0$ and $\dot{x}(0) = 2$, solve the equation, and find the greatest displacement.

Question 18

Question 18 – Using an integrating factor with integration *(Section C · Problem solving)*

Consider the differential equation $\dfrac{dy}{dx} + y = xe^{-x}$, with $y = 1$ when $x = 0$.

a. Find the integrating factor and show that $\dfrac{d}{dx}(ye^{x}) = x$.
b. Find y in terms of x.
c. Show that $\dfrac{dy}{dx} = -\frac{1}{2}(x-1)^2 e^{-x}$.
d. Hence describe the shape of the curve for $x \geq 0$, including the nature of the point where $x = 1$.

Question 19

Question 19 – Two linked quantities *(Section C · Problem solving)*

Two quantities x and y vary with time t according to $\dfrac{dx}{dt} = -x + 2y$ and $\dfrac{dy}{dt} = -2x - y$, with $x = 1$ and $y = 0$ at $t = 0$.

a. Show that $y = \frac{1}{2}\left(\dfrac{dx}{dt} + x\right)$ and use it to show that $\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + 5x = 0$.
b. Solve for x in terms of t.
c. Find y in terms of t.
d. Describe how the point (x, y) moves in the plane as t increases.

Question 20

Question 20 – Challenge: energy in oscillations *(Section D · Challenge)*

A particle of unit mass satisfies $\ddot{x} + 2\lambda\dot{x} + \omega^2 x = 0$ with $\lambda \geq 0$. Its energy is

$E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2$.

a. Show that $\dfrac{dE}{dt} = -2\lambda\dot{x}^2$.
b. Deduce that E is constant when $\lambda = 0$ and decreasing when $\lambda > 0$.
c. For $\lambda = 0$ and $x = 3\cos 4t$, verify that E is constant and find its value.
d. For the scooter in Question 11, $x = e^{-2t}\left(\frac{1}{2}\cos 3t + \frac{1}{3}\sin 3t\right)$ with $\lambda = 2$ and $\omega^2 = 13$. Verify numerically that $\dfrac{dE}{dt} = -4\dot{x}^2$ at $t = 1$.