Answers - Differential equations

Year: Further Maths

Answer to Question 1

a. $I = e^{3x}$. $\frac{d}{dx}(ye^{3x}) = 6e^{3x}$, so $ye^{3x} = 2e^{3x} + A$ and $y = 2 + Ae^{-3x}$.

b. Divide by x: $\frac{dy}{dx} + \frac{2}{x}y = 4$. $I = e^{2\ln x} = x^2$. $\frac{d}{dx}(x^2y) = 4x^2$, so $x^2y = \frac{4}{3}x^3 + C$ and $y = \frac{4}{3}x + \frac{C}{x^2}$.

c. $I = e^{\int \tan x\,dx} = e^{-\ln\cos x} = \sec x$. $\frac{d}{dx}(y\sec x) = \cos x \sec x = 1$, so $y\sec x = x + C$ and $y = (x + C)\cos x$.

d. $I = e^{2x}$. $\frac{d}{dx}(ye^{2x}) = e^x$, so $ye^{2x} = e^x + C$ and $y = e^{-x} + Ce^{-2x}$. When $x = 0$: $1 + C = 3$, $C = 2$. So $y = e^{-x} + 2e^{-2x}$.

Answer to Question 2

Method: for $ay'' + by' + cy = 0$, solve the auxiliary equation $am^2 + bm + c = 0$. Two real roots give $Ae^{m_1x} + Be^{m_2x}$; a repeated root gives $(A + Bx)e^{mx}$; complex roots $p \pm qi$ give $e^{px}(A\cos qx + B\sin qx)$.

a. The auxiliary equation factorises. $m^2 - 5m + 6 = (m - 2)(m - 3) = 0$, distinct real roots 2 and 3: $y = Ae^{2x} + Be^{3x}$ Check: e^{2x} gives $4 - 10 + 6 = 0$.

b. The auxiliary equation is a perfect square. $m^2 + 6m + 9 = (m + 3)^2 = 0$, repeated root -3: $y = (A + Bx)e^{-3x}$ The factor x is needed because $Ae^{-3x} + Be^{-3x}$ would contain only one constant.

c. The discriminant is negative, so the roots are complex with $p = -2$ and $q = 3$. $m = \frac{-4 \pm \sqrt{16 - 52}}{2} = -2 \pm 3i$: $y = e^{-2x}(A\cos 3x + B\sin 3x)$ The solution oscillates with decreasing amplitude.

d. Here the roots are purely imaginary. $m^2 + 16 = 0$, $m = \pm 4i$ (so $p = 0$): $y = A\cos 4x + B\sin 4x$ This is undamped oscillation, as in simple harmonic motion.

Answer to Question 3

a. Try $y = ax + b$: $-4a + 3ax + 3b = 6x$, so $a = 2$ and $3b = 8$, $b = \frac{8}{3}$. The CF has roots 1 and 3. General solution $y = Ae^x + Be^{3x} + 2x + \frac{8}{3}$.

b. The roots of $m^2 + m - 6 = 0$ are 2 and -3, so e^{2x} is in the CF. Try $y = kxe^{2x}$:

$y' = k(1+2x)e^{2x}$, $y'' = k(4+4x)e^{2x}$. Substituting, $k(4+4x) + k(1+2x) - 6kx = 5k$, so $5k = 10$, $k = 2$. General solution $y = Ae^{2x} + Be^{-3x} + 2xe^{2x}$.

c. Try $y = a\sin 2x + b\cos 2x$. $y'' = -4y$, so $y'' + 9y = 5y$, giving $5a = 1$ and $b = 0$. The CF is $A\cos 3x + B\sin 3x$. General solution $y = A\cos 3x + B\sin 3x + \frac{1}{5}\sin 2x$.

Answer to Question 4

Method: for $ay'' + by' + cy = 0$, solve the auxiliary equation $am^2 + bm + c = 0$. Two real roots give $Ae^{m_1 x} + Be^{m_2 x}$; a repeated root gives $(A + Bx)e^{mx}$; complex roots $p \pm qi$ give $e^{px}(A\cos qx + B\sin qx)$.

a. $m^2 - m - 2 = (m-2)(m+1) = 0$, so $m = 2$ or $m = -1$.

b. Two distinct real roots give a sum of two exponentials. $y = Ae^{2x} + Be^{-x}$.

c. Differentiate the general solution before substituting $x = 0$ into y'. $y(0) = A + B = 1$. $y' = 2Ae^{2x} - Be^{-x}$, so $y'(0) = 2A - B = 5$. Adding: $3A = 6$, $A = 2$, $B = -1$. $y = 2e^{2x} - e^{-x}$. Check: $y(0) = 2 - 1 = 1$ and $y'(0) = 4 + 1 = 5$.

d. Use $e^{2\ln 2} = 4$ and $e^{-\ln 2} = \frac{1}{2}$. $y = 2 \cdot 4 - \frac{1}{2} = 7.5$ So $y(\ln 2) = 7.5$.

Answer to Question 5

a. $m^2 + 16 = 0$, so $x = A\cos 4t + B\sin 4t$. $x(0) = A = 3$ and $x'(0) = 4B = 0$. So $x = 3\cos 4t$.

b. Amplitude **3 m**; period $\dfrac{2\pi}{4} = \dfrac{\pi}{2} \approx 1.57$ **s**.

c. $\dot{x} = -12\sin 4t$, so the greatest speed is **12 m s^{-1}** (when passing through the centre).

d. $v^2 = \omega^2(a^2 - x^2) = 16(9 - 4) = 80$, so $v = 4\sqrt{5} \approx 8.94$ **m s^{-1}**.

Answer to Question 6

a. $y' = -2e^{-2x}$, $y'' = 4e^{-2x}$. Then $4e^{-2x} - 8e^{-2x} + 4e^{-2x} = \mathbf{0}$.

b. $y' = (1-2x)e^{-2x}$, $y'' = (4x-4)e^{-2x}$. Then $(4x-4) + 4(1-2x) + 4x = 0$, so **it is a solution**.

c. The auxiliary equation has the repeated root -2, so $y = (A + Bx)e^{-2x}$. A second-order equation needs two independent solutions to fit two initial conditions.

d. $y(0) = A = 1$. $y' = (B - 2A - 2Bx)e^{-2x}$, so $y'(0) = B - 2 = 0$, $B = 2$. $y = (1 + 2x)e^{-2x}$.

Answer to Question 7

Method: the substitution $\theta = T - 20$ turns the law of cooling into $\frac{d\theta}{dt} = -k\theta$, whose solution is an exponential decay. Use the two data points to find the constant and k.

a. Separate variables or use $\theta = T - 20$: $\frac{d\theta}{dt} = -k\theta$, so $\theta = \theta_0 e^{-kt}$. With $\theta_0 = 70$: $T = 20 + 70e^{-kt}$.
b. $60 = 20 + 70e^{-5k}$ gives $e^{-5k} = \frac{4}{7}$, so $5k = \ln\frac{7}{4}$ and $k = \frac{1}{5}\ln\frac{7}{4} =$ **0.112** (3 s.f.).
c. $40 = 20 + 70e^{-kt}$, $e^{-kt} = \frac{2}{7}$, $t = \frac{\ln 3.5}{k} = \frac{1.2528}{0.11192} \approx$ **11.2 minutes**.
d. $e^{-kt} \to 0$, so $T \to 20°$**C**, the room temperature.

Answer to Question 8

Method (integrating factor): write the equation as $\frac{dy}{dx} + Py = Q$, multiply by $I = e^{\int P\,dx}$; the left-hand side becomes $\frac{d}{dx}(Iy)$. Integrate both sides, remember the constant, then divide by I.

a. Move the C term to the left; $P = 0.2$, so $I = e^{\int 0.2\,dt}$. $\frac{dC}{dt} + 0.2C = 4$. $I = e^{0.2t}$.
b. Multiply through by I and integrate: $\int 4e^{0.2t}dt = 20e^{0.2t}$. $\frac{d}{dt}(Ce^{0.2t}) = 4e^{0.2t}$, so $Ce^{0.2t} = 20e^{0.2t} + K$. With $C(0) = 0$: $K = -20$. So $C = 20(1 - e^{-0.2t})$. This is the required result.
c. Look at the exponential term as t grows. $e^{-0.2t} \to 0$, so $C \to$ **20 mg per litre**. This is where the drip rate balances the rate at which the body removes the drug.
d. Set $C = 15$ and solve with logarithms. $15 = 20(1 - e^{-0.2t})$, $e^{-0.2t} = \frac{1}{4}$, $t = 5\ln 4 =$ **6.93 hours**. So the target concentration is reached after about 6.93 hours.

Answer to Question 9

a. Rate in $= 0.1 \times 4 = 0.4$ kg per minute. Concentration in the tank is $\frac{Q}{200}$, so rate out $= 4 \times \frac{Q}{200} = \frac{Q}{50}$. Net rate: $\frac{dQ}{dt} = 0.4 - \frac{Q}{50}$.
b. $\frac{dQ}{dt} + \frac{Q}{50} = 0.4$, $I = e^{t/50}$, $Qe^{t/50} = 20e^{t/50} + K$. $Q(0) = 10$ gives $K = -10$: $Q = 20 - 10e^{-t/50}$.
c. $15 = 20 - 10e^{-t/50}$, $e^{-t/50} = \frac{1}{2}$, $t = 50\ln 2 =$ **34.7 minutes**.
d. $Q \to$ **20 kg**, which is 0.1×200: the tank ends up with the same concentration (0.1 kg per litre) as the incoming brine.

Answer to Question 10

a. $RC = 2000 \times 500 \times 10^{-6} = 1$ second. So $\dfrac{dV}{dt} + V = 12$.

b. $I = e^t$, $Ve^t = 12e^t + K$, $V(0) = 0$ gives $K = -12$: $V = 12(1 - e^{-t})$.

c. $9 = 12(1 - e^{-t})$, $e^{-t} = \frac{1}{4}$, $t = \ln 4 \approx$ **1.39 s**.

d. $\dfrac{dV}{dt} = 12e^{-t}$, which equals **12 V s**$^{-1}$ at $t = 0$: the voltage rises fastest at the start and slows as the capacitor fills.

Answer to Question 11

a. $2m^2 + 8m + 26 = 0$, so $m^2 + 4m + 13 = 0$, $m = \dfrac{-4 \pm \sqrt{16 - 52}}{2} = -2 \pm 3i$. The roots are complex, so the damping is **light** (underdamped).

b. $x = e^{-2t}(A\cos 3t + B\sin 3t)$. $x(0) = A = \frac{1}{2}$.
$\dot{x} = e^{-2t}\left[(-2A + 3B)\cos 3t + (-2B - 3A)\sin 3t\right]$ and $\dot{x}(0) = -2A + 3B = 0$, so $B = \frac{1}{3}$. $x = e^{-2t}\left(\frac{1}{2}\cos 3t + \frac{1}{3}\sin 3t\right)$.

c. Period $= \dfrac{2\pi}{3} \approx$ **2.09 s**. Each period multiplies the amplitude by $e^{-2\times 2\pi/3} = e^{-4\pi/3} \approx$ **0.0152**.

d. With $c = 20$: $m^2 + 10m + 13 = 0$, discriminant $100 - 52 = 48 > 0$, two distinct negative real roots: the damping is **heavy** and the scooter returns to equilibrium without oscillating.

Answer to Question 12

a. Set $\dfrac{dA}{dt} = 0$: $A = \dfrac{2000}{0.04} =$ **£50 000**.

b. $\dfrac{dA}{dt} - 0.04A = -2000$, $I = e^{-0.04t}$, $Ae^{-0.04t} = 50000e^{-0.04t} + K$, so $A = 50000 + Ke^{0.04t}$. $A(0) = 40000$ gives $K = -10000$. $A = 50000 - 10000e^{0.04t}$.

c. $A = 0$ when $e^{0.04t} = 5$, $t = 25\ln 5 \approx$ **40.2 years**.

d. **£50 000**, since then $A' = 0$ at all times. Starting below it, the balance falls ever faster.

Answer to Question 13

a. $m^2 + 4 = 0$ gives $m = \pm 2i$: $x_{CF} = A\cos 2t + B\sin 2t$.

b. $x'' = -9a\cos 3t$, so $-9a + 4a = 10$, $a = -2$. General solution $x = A\cos 2t + B\sin 2t - 2\cos 3t$.

c. The forcing has the same frequency as the CF, so $a\cos 2t + b\sin 2t$ gives zero on the left side. For $x = 2t\sin 2t$: $\dot{x} = 2\sin 2t + 4t\cos 2t$ and $\ddot{x} = 8\cos 2t - 8t\sin 2t$. Then $\ddot{x} + 4x = 8\cos 2t$. ✓.

d. The factor t means the amplitude grows without limit: **resonance**. If walkers match the natural frequency of the bridge, the swaying increases steadily, so soldiers break step to avoid that.

Answer to Question 14

Method (integrating factor): write the equation as $\frac{dy}{dx} + Py = Q$, multiply by $I = e^{\int P\,dx}$; the left-hand side becomes $\frac{d}{dx}(Iy)$. Integrate both sides, remember the constant, then divide by I.

a. At terminal speed the acceleration is zero. Terminal speed when $\frac{dv}{dt} = 0$: $v = \frac{9.8}{0.2} =$ **49 m s^{-1}**. This is where air resistance balances the weight.

b. $\frac{dv}{dt} + 0.2v = 9.8$, $I = e^{0.2t}$, $ve^{0.2t} = 49e^{0.2t} + K$, $K = -49$ (since $v(0) = 0$). So $v = 49(1 - e^{-0.2t})$. This is the required result.

c. Substitute $t = 10$. $v(10) = 49(1 - e^{-2}) = 49 \times 0.8647 =$ **42.4 m s^{-1}**. So after 10 s the skydiver is at about 87% of terminal speed.

d. Distance is the integral of speed. $s = \int_0^{10} 49(1 - e^{-0.2t})\,dt = 49\left[t + 5e^{-0.2t}\right]_0^{10} = 49(10 + 5e^{-2} - 5) =$ **278 m** (3 s.f.). So the skydiver falls about 278 m in the first 10 seconds.

Answer to Question 15

Method (integrating factor): write the equation as $\frac{dy}{dx} + Py = Q$, multiply by $I = e^{\int P\,dx}$; the left-hand side becomes $\frac{d}{dx}(Iy)$. Integrate both sides, remember the constant, then divide by I.

a. Divide through by x so that the coefficient of $\frac{dy}{dx}$ is 1; then $P = -\frac{2}{x}$. $\frac{dy}{dx} - \frac{2}{x}y = x^3$. $I = e^{-2\ln x} = x^{-2}$. Using $e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2}$.

b. Multiply by x^{-2}; the left-hand side is the derivative of $\frac{y}{x^2}$. $\frac{d}{dx}\left(\frac{y}{x^2}\right) = x$, so $\frac{y}{x^2} = \frac{x^2}{2} + C$: $y = \frac{x^4}{2} + Cx^2$. Check: $xy' - 2y = x(2x^3 + 2Cx) - 2\left(\frac{x^4}{2} + Cx^2\right) = x^4$.

c. Substitute $x = 1$, $y = 3$. $3 = \frac{1}{2} + C$, $C = \frac{5}{2}$: $y = \frac{x^4}{2} + \frac{5}{2}x^2$.

d. $y(2) = 8 + 10 = \mathbf{18}$. $y' = 2x^3 + 5x > 0$ for $x > 0$, so there is no stationary point. So the curve is always increasing for $x > 0$.

Answer to Question 16

a. $m^2 + 2m + 5 = 0$, $m = -1 \pm 2i$: $y_{CF} = e^{-x}(A\cos 2x + B\sin 2x)$.

b. $y' = a$, $y'' = 0$: $2a + 5ax + 5b = 5x + 12$. So $a = 1$ and $2 + 5b = 12$, $b = 2$. $y_{PI} = x + 2$.

c. $y = e^{-x}(A\cos 2x + B\sin 2x) + x + 2$

d. $y(0) = A + 2 = 3$, $A = 1$. $y' = e^{-x}[(-A + 2B)\cos 2x + (-B - 2A)\sin 2x] + 1$, and $y'(0) = -A + 2B + 1 = 0$ gives $B = 0$. So $y = e^{-x}\cos 2x + x + 2$. As x grows the oscillating term dies away and $y \approx x + 2$.

Answer to Question 17

a. Critical damping when the discriminant is zero: $c^2 - 100 = 0$, so $c = 10$.

b. $m^2 + 6m + 25 = 0$, $m = -3 \pm 4i$. The roots are complex, so the motion **oscillates** with decaying amplitude.

c. $m^2 + 12m + 25 = 0$, discriminant $144 - 100 = 44 > 0$, $m = -6 \pm \sqrt{11}$. $x = Ae^{(-6+\sqrt{11})t} + Be^{(-6-\sqrt{11})t}$.

d. $x = (A + Bt)e^{-5t}$. $x(0) = A = 0$, $\dot{x} = Be^{-5t}(1 - 5t)$, $\dot{x}(0) = B = 2$. $x = 2te^{-5t}$. $\dot{x} = 0$ at $t = 0.2$, so the greatest displacement is $0.4e^{-1} \approx$ **0.147 m**.

Answer to Question 18

a. $I = e^x$. Multiplying: $e^x\frac{dy}{dx} + ye^x = x$, which is $\frac{d}{dx}(ye^x) = x$.

b. $ye^x = \frac{1}{2}x^2 + C$. $y(0) = 1$ gives $C = 1$. $y = \left(\frac{1}{2}x^2 + 1\right)e^{-x}$.

c. Product rule: $\frac{dy}{dx} = xe^{-x} - \left(\frac{1}{2}x^2 + 1\right)e^{-x} = e^{-x}\left(x - \frac{1}{2}x^2 - 1\right) = -\frac{1}{2}(x - 1)^2e^{-x}$.

d. The gradient is never positive and is zero only at $x = 1$. The curve **decreases for all** $x \geq 0$ and has a stationary point of inflection at $x = 1$, with $y = 1.5e^{-1} \approx 0.552$. It tends to 0 as $x \to \infty$.

Answer to Question 19

a. From the first equation $2y = \dot{x} + x$. Differentiate: $2\dot{y} = \ddot{x} + \dot{x}$. Substitute $\dot{y} = -2x - y$: $\ddot{x} + \dot{x} = -4x - 2y = -4x - (\dot{x} + x)$, so $\ddot{x} + 2\dot{x} + 5x = 0$.

b. $m = -1 \pm 2i$, $x = e^{-t}(A\cos 2t + B\sin 2t)$. $x(0) = 1$ gives $A = 1$. $\dot{x}(0) = -x(0) + 2y(0) = -1$ and $\dot{x}(0) = -A + 2B$, so $B = 0$. $x = e^{-t}\cos 2t$.
c. $\dot{x} = -e^{-t}\cos 2t - 2e^{-t}\sin 2t$, so $y = \frac{1}{2}(\dot{x} + x) = -e^{-t}\sin 2t$.
d. $x^2 + y^2 = e^{-2t}$, so the point stays on a circle of radius e^{-t} that shrinks: it **spirals clockwise into the origin** (at $t = \frac{\pi}{4}$ it is at $-e^{-\pi/4}$ on the y-axis).

Answer to Question 20

a. $\dfrac{dE}{dt} = \dot{x}\ddot{x} + \omega^2 x\dot{x} = \dot{x}(\ddot{x} + \omega^2 x) = \dot{x}(-2\lambda\dot{x}) = -2\lambda\dot{x}^2$.
b. If $\lambda = 0$ then $\dfrac{dE}{dt} = 0$, so E is constant. If $\lambda > 0$ then $\dfrac{dE}{dt} \leq 0$ and the energy falls (friction removes it).
c. $\dot{x} = -12\sin 4t$, so $E = 72\sin^2 4t + \frac{1}{2}\cdot 16 \cdot 9\cos^2 4t = 72\sin^2 4t + 72\cos^2 4t = \mathbf{72}$, constant.
d. Here $\ddot{x} + 4\dot{x} + 13x = 0$ (the scooter equation divided by 2), so $2\lambda = 4$. Evaluate $\dot{x}(1)$ and $E'(1)$ numerically: both sides equal $-4\dot{x}(1)^2 \approx -0.0068$. ✓