Test answers - De Moivre's theorem and roots of unity

Year: Further Maths · Duration: 60 minutes · /20

Answer to Question 1 (4 marks)

Method: write the number in modulus-argument form $r(\cos\theta + i\sin\theta)$, then use de Moivre: $[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$. Reduce the angle $n\theta$ by multiples of 2π before evaluating. Mark scheme as shown in brackets.

a. $1 - i$ is in the fourth quadrant, at angle $\frac{\pi}{4}$ below the real axis. $|1 - i| = \sqrt{2}$, $\arg = -\frac{\pi}{4}$ [1]: $\sqrt{2}\left(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4})\right)$ [1]

b. Raise the modulus to the power 10 and multiply the argument by 10. $(\sqrt{2})^{10} = 32$ [1] and angle $-\frac{10\pi}{4} = -\frac{5\pi}{2} \equiv -\frac{\pi}{2}$, so $32(\cos(-\frac{\pi}{2}) + i\sin(-\frac{\pi}{2})) = -32i$ [1] Since $-\frac{5\pi}{2} + 2\pi = -\frac{\pi}{2}$, the answer is $-32i$.

Answer to Question 2 (5 marks)

Method: with $z = \cos\theta + i\sin\theta$, use $z^n + \dfrac{1}{z^n} = 2\cos n\theta$ and $z^n - \dfrac{1}{z^n} = 2i\sin n\theta$.

Expand with the binomial theorem, pair the terms z^k and z^{-k}, and compare with the power of $2\cos\theta$ or $2i\sin\theta$. Mark scheme as shown in brackets.

a. $\left(z - \frac{1}{z}\right)^4 = z^4 - 4z^2 + 6 - 4z^{-2} + z^{-4}$ [1]. This is $2\cos 4\theta - 8\cos 2\theta + 6$ [1]. The left side is $16\sin^4\theta$, giving the result [1].

b. $\frac{1}{8}\left[\frac{\sin 4\theta}{4} - 2\sin 2\theta + 3\theta\right]_0^{\pi/4}$ [1] $= \frac{1}{8}\left(0 - 2 + \frac{3\pi}{4}\right) = \dfrac{3\pi - 8}{32}$ [1]

Answer: $\displaystyle\int_0^{\pi/4} \sin^4\theta\, d\theta = \frac{3\pi - 8}{32} \approx 0.0445.$

In (b), the term $\frac{\sin 4\theta}{4}$ vanishes at both limits because $\sin\pi = 0$.

Answer to Question 3 (5 marks)

Method: write the right-hand side in modulus-argument form, take the cube root of the modulus, and add multiples of $\frac{2\pi}{3}$ to the argument to get all three roots. Mark scheme as shown in brackets.

a. $-27i = 27\left(\cos(-\frac{\pi}{2}) + i\sin(-\frac{\pi}{2})\right)$, so $r = 3$ [1] and $\theta = -\frac{\pi}{6} + \frac{2\pi k}{3}$ [1]: $3\left(\cos(-\frac{\pi}{6}) + i\sin(-\frac{\pi}{6})\right)$, $3\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$, $3\left(\cos(-\frac{5\pi}{6}) + i\sin(-\frac{5\pi}{6})\right)$ [1]

b. Equilateral triangle on a circle of radius 3 [1]. Area

$= \frac{1}{2} \cdot 3 \cdot 3 \cdot \sin 120^\circ \times 3 = \frac{27\sqrt{3}}{4} \approx$ **11.7** [1]

The three roots are equally spaced at 120° on the circle of radius 3, so they form an equilateral triangle.

Answer to Question 4 (3 marks)

Method: replace $\cos 3\theta$ by $4c^3 - 3c$ where $c = \cos\theta$, collect on one side and factorise completely. Then solve each factor in the interval $0 \leq \theta \leq \pi$. Mark scheme as shown in brackets.

a. $4c^3 - 3c = c$ gives $4c^3 - 4c = 0$ [1], $4c(c-1)(c+1) = 0$ [1]. So $\cos\theta = 0, 1, -1$ and $\theta = 0,\ \frac{\pi}{2},\ \pi$ [1]

Check: $\cos\frac{3\pi}{2} = 0 = \cos\frac{\pi}{2}$, $\cos 0 = 1$ and $\cos 3\pi = -1 = \cos\pi$.

The common error is dividing both sides by $\cos\theta$, which loses $\theta = \frac{\pi}{2}$. There are no other solutions, because each of $\cos\theta = 0$, 1 and -1 has exactly one solution in $[0, \pi]$.

Answer to Question 5 (3 marks)

Method: the nth roots of unity are $e^{2k\pi i/n}$ for $k = 0, 1, \ldots, n-1$; scaling by 5 multiplies every modulus by 5. Mark scheme as shown in brackets.

a. For sixth roots, the angles are multiples of $\frac{2\pi}{6} = \frac{\pi}{3}$. $5e^{ik\pi/3}$, $k = 0, \ldots, 5$ [1]
b. Adjacent corners and the centre form an equilateral triangle, since the angle at the centre is $\frac{\pi}{3}$. The hexagon is made of six such triangles. The side equals the radius: **5 cm** [1]. Area $= 6 \times \frac{1}{2} \times 5 \times 5 \times \sin 60^\circ = \frac{75\sqrt{3}}{2} \approx$ **65.0 cm**^{2} [1]