Lesson - De Moivre's theorem and roots of unity

Year: Further Maths

De Moivre's theorem turns powers and roots of complex numbers into simple operations on the modulus and argument. It also gives multiple-angle formulae and the geometry of roots of unity.

Starter activity

Imagine a spinner on a table. Its pointer is one unit long and starts pointing along the positive real axis. You turn it through 40°, then through another 40°, and again until you have turned it through 360°. How many turns of 40° did it take? What complex number does the pointer represent after each turn? If one turn is the multiplication by $\cos 40^\circ + i \sin 40^\circ$, what do you think happens when you multiply by this number nine times? Write down your conjecture before reading on.

De Moivre's theorem and Euler's form

Theorem: de Moivre's theorem

For any integer n,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

So if $z = r(\cos\theta + i\sin\theta)$ then $z^n = r^n(\cos n\theta + i\sin n\theta)$. It is proved by induction for positive n, and extended to negative n using $z^{-1} = \frac{1}{r}(\cos\theta - i\sin\theta)$.

Definition: Euler's form

$e^{i\theta} = \cos\theta + i\sin\theta$, so $z = re^{i\theta}$ and $z^n = r^n e^{in\theta}$. Also $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ and $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$.

Key fact: unit modulus

If $z = e^{i\theta}$ then $z^n + z^{-n} = 2\cos n\theta$ and $z^n - z^{-n} = 2i\sin n\theta$.

Example 1

Evaluate $(1+i)^{10}$.

$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$, so

$(1+i)^{10} = (\sqrt{2})^{10}\left(\cos\frac{10\pi}{4} + i\sin\frac{10\pi}{4}\right) = 32\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 32i.$

Watch out

De Moivre only applies to $\cos\theta + i\sin\theta$. If the modulus is not 1, raise the modulus to the power n as well.

Multiple-angle and power formulae

Method: multiple angles

1. Write $c = \cos\theta$, $s = \sin\theta$ and expand $(c+is)^n$ with the binomial theorem.
2. Use de Moivre: the expansion equals $\cos n\theta + i\sin n\theta$.
3. Equate real parts for $\cos n\theta$ and imaginary parts for $\sin n\theta$.
4. Replace s^2 by $1-c^2$ (or c^2 by $1-s^2$) to use a single function.

Example 2

Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

$(c+is)^3 = c^3 + 3ic^2s - 3cs^2 - is^3$. The real part gives $\cos 3\theta = c^3 - 3cs^2 = c^3 - 3c(1-c^2) = 4c^3 - 3c$.

Method: expressing powers of $\cos\theta$ as multiple angles

1. Let $z = \cos\theta + i\sin\theta$, so $2\cos\theta = z + z^{-1}$ (or $2i\sin\theta = z - z^{-1}$).
2. Expand $(z+z^{-1})^n$ with the binomial theorem.
3. Group terms into pairs $z^k + z^{-k} = 2\cos k\theta$.
4. Divide by 2^n.

Example 3

Express $\cos^4\theta$ in terms of multiple angles.

$(2\cos\theta)^4 = z^4 + 4z^2 + 6 + 4z^{-2} + z^{-4} = 2\cos 4\theta + 8\cos 2\theta + 6$. Hence $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$. This makes integrals of $\cos^n\theta$ easy.

nth roots of a complex number

Rule: nth roots

To solve $z^n = w = R(\cos\alpha + i\sin\alpha)$, write w with the general argument $\alpha + 2\pi k$. Then

$$z = R^{1/n}\left[\cos\frac{\alpha+2\pi k}{n} + i\sin\frac{\alpha+2\pi k}{n}\right], \quad k = 0, 1, \ldots, n-1.$$

The n roots have the same modulus and are equally spaced by $\frac{2\pi}{n}$, so they form a regular n-gon centred at the origin.

Example 4

Solve $z^3 = 8$.

$8 = 8(\cos 2\pi k + i\sin 2\pi k)$, so $z = 2\left(\cos\frac{2\pi k}{3} + i\sin\frac{2\pi k}{3}\right)$, giving 2, $-1+\sqrt{3}i$ and $-1-\sqrt{3}i$.

Example 5

Solve $z^4 = 1 - i$.

$|1-i| = \sqrt{2}$ and $\arg(1-i) = -\frac{\pi}{4}$. So $z = 2^{1/8}\left[\cos\left(-\frac{\pi}{16} + \frac{k\pi}{2}\right) + i\sin\left(-\frac{\pi}{16} + \frac{k\pi}{2}\right)\right]$, $k = 0, 1, 2, 3$. The four roots lie on a circle of radius $2^{1/8}$, each 90° from the next.

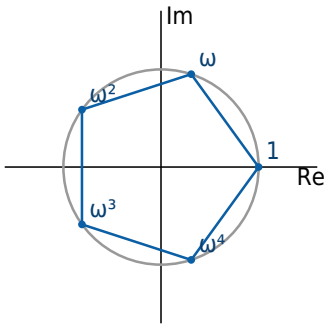
Watch out

If you forget the $2\pi k$ you find only one root. Remember also to take the nth root of the modulus, and give the arguments in the principal range if the question asks.

Roots of unity

Definition: roots of unity

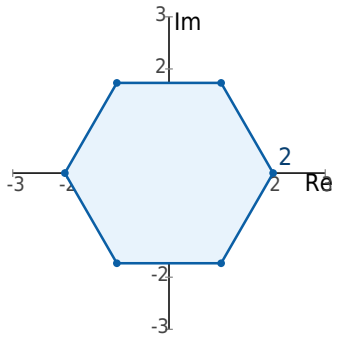
The solutions of $z^n = 1$ are $e^{2\pi i k/n}$, $k = 0, \ldots, n-1$. Writing $\omega = e^{2\pi i/n}$, they are $1, \omega, \omega^2, \ldots, \omega^{n-1}$.



The fifth roots of unity lie on the unit circle at angles 0, 72, 144, 216, 288 degrees.

Key fact: sum and product

For $n \geq 2$, $1 + \omega + \cdots + \omega^{n-1} = \dfrac{\omega^n - 1}{\omega - 1} = 0$; in particular $1 + \omega + \omega^2 = 0$ for cube roots. The product of the roots is $(-1)^{n+1}$. Taking real parts of the sum gives results such as $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$.



The six roots of z^6 = 64 form a regular hexagon of circumradius 2.

Note: area of the polygon

The nth roots of w are the vertices of a regular polygon. If the circumradius is r, its area is $\frac{n}{2} r^2 \sin\frac{2\pi}{n}$.

Example 6

Show that the sixth roots of unity sum to zero.

With $\omega = e^{i\pi/3}$, the real parts are $1, \frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2}$, and $1 + \frac{1}{2} - \frac{1}{2} - 1 - \frac{1}{2} + \frac{1}{2} = 0$. The imaginary parts cancel in pairs too.

Worked example 5: a high power

Find $\left(\dfrac{1 + i\sqrt{3}}{2}\right)^{100}$.

The number inside the bracket has modulus $\sqrt{\frac{1}{4} + \frac{3}{4}} = 1$ and argument $\frac{\pi}{3}$, so it equals $\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$. De Moivre's theorem gives $\cos\frac{100\pi}{3} + i\sin\frac{100\pi}{3}$. Now $\frac{100\pi}{3} = 33\pi + \frac{\pi}{3}$, and since 32π is a whole number of turns, this is equivalent to $\pi + \frac{\pi}{3} = \frac{4\pi}{3}$. The answer is $\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$. The skill here is to reduce the angle by removing multiples of 2π before evaluating.

Note: what Euler's formula adds

De Moivre's theorem is neat but it is really a shadow of a deeper fact. If we agree that $e^{i\theta}$ means $\cos\theta + i\sin\theta$, then the rule $\left(e^{i\theta}\right)^n = e^{in\theta}$ is just the familiar law of indices, and the product rule $e^{i\alpha}e^{i\beta} = e^{i(\alpha+\beta)}$ contains the addition formulae of trigonometry.

This is why working with exponentials is so efficient: instead of remembering long lists of identities, you use the laws of indices and read off the real and imaginary parts at the end. In this chapter you will use the form $z = re^{i\theta}$ for products, powers and roots, and the form $\cos\theta + i\sin\theta$ when you need to expand or compare real and imaginary parts.

Method: expressing $\cos n\theta$ or $\sin n\theta$ in powers

1. Write $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ using de Moivre's theorem.
2. Expand the left-hand side using the binomial theorem and the fact that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$.
3. Equate the real parts to obtain $\cos n\theta$, or the imaginary parts to obtain $\sin n\theta$.
4. Replace even powers of the other function using $\sin^2\theta = 1 - \cos^2\theta$ (or the reverse) to obtain a polynomial in a single function.

Worked example 6: $\sin 5\theta$ in terms of $\sin\theta$

Show that $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$.

Write $c = \cos\theta$ and $s = \sin\theta$. The binomial expansion of $(c + is)^5$ has imaginary part $5c^4 s - 10c^2 s^3 + s^5$, since the terms with odd powers of i contribute i, $-i$ and i respectively. Replace c^2 by $1 - s^2$: the first term becomes $5(1-s^2)^2 s = 5s - 10s^3 + 5s^5$, the second becomes $-10(1-s^2)s^3 = -10s^3 + 10s^5$, and the last is s^5. Adding gives $5s - 20s^3 + 16s^5$, which is the required result. A quick check at $\theta = \frac{\pi}{2}$ gives $\sin\frac{5\pi}{2} = 1$ and $16 - 20 + 5 = 1$.

Method: finding the nth roots of a complex number

1. Write the number in modulus-argument form as $r(\cos\alpha + i\sin\alpha)$ and add $2\pi k$ to the argument.
2. Take the nth root of the modulus, giving $r^{1/n}$, and divide the argument by n.
3. Write $z_k = r^{1/n}\left(\cos\frac{\alpha+2\pi k}{n} + i\sin\frac{\alpha+2\pi k}{n}\right)$ for $k = 0, 1, \ldots, n-1$.
4. Check that exactly n distinct roots have been found, equally spaced by $\frac{2\pi}{n}$ round a circle of radius $r^{1/n}$, and convert to Cartesian form if asked.

Worked example 7: solving a quartic equation

Solve $z^4 = -4$ and use the answer to factorise $z^4 + 4$ into two real quadratics.

We have $-4 = 4(\cos\pi + i\sin\pi)$, so $|z| = 4^{1/4} = \sqrt{2}$ and the arguments are $\frac{\pi+2\pi k}{4}$, namely $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$. The roots are $1+i$, $-1+i$, $-1-i$ and $1-i$. The conjugate pair $1 \pm i$ gives the factor $z^2 - 2z + 2$, and the pair $-1 \pm i$ gives $z^2 + 2z + 2$. Multiplying them, $(z^2+2)^2 - 4z^2 = z^4 + 4z^2 + 4 - 4z^2 = z^4 + 4$, so $z^4 + 4 = (z^2 - 2z + 2)(z^2 + 2z + 2)$. This is a famous factorisation that cannot be found

by inspection.

Key fact: the geometry of roots of unity

The nth roots of unity are the vertices of a regular n-sided polygon inscribed in the unit circle, with one vertex at the point 1. If $\omega = e^{2\pi i/n}$, the roots are $1, \omega, \omega^2, \ldots, \omega^{n-1}$, and each root is obtained from the previous one by a rotation through $\frac{2\pi}{n}$. Their sum is zero because the polygon is balanced about the origin, and their product is $(-1)^{n+1}$. Every root other than 1 satisfies $1 + z + z^2 + \cdots + z^{n-1} = 0$. The roots also come in conjugate pairs, since , and this is the reason why $z^n - 1$ factorises into real linear and quadratic factors.

Watch out: common errors

- De Moivre's theorem applies only when the modulus is 1 inside the bracket. For $(1+i)^8$ you must also raise the modulus $\sqrt{2}$ to the power 8.
- When finding roots, do not forget the $2\pi k$. Writing only the first root loses the others, and each equation $z^n = w$ has exactly n solutions.
- Divide the whole argument by n, including the $2\pi k$ term, not only the principal argument.
- When expanding $\left(z + \frac{1}{z}\right)^n$, use $z^n + z^{-n} = 2\cos n\theta$ and $z^n - z^{-n} = 2i\sin n\theta$, and remember that $\left(z - \frac{1}{z}\right)^n = (2i\sin\theta)^n$ contains a factor i^n.

Worked example 8: a geometric problem with roots

The points A, B, C represent the cube roots of 27 on an Argand diagram. Find the area of triangle ABC and the length of each side.

The roots of $z^3 = 27$ are 3, $3e^{2\pi i/3}$ and $3e^{4\pi i/3}$. They lie on a circle of radius 3 and are equally spaced, so the triangle is equilateral. The distance between 3 and $3e^{2\pi i/3}$ is $3|1 - e^{2\pi i/3}| = 3 \times 2\sin\frac{\pi}{3} = 3\sqrt{3}$. The area of an equilateral triangle with side $3\sqrt{3}$ is $\frac{\sqrt{3}}{4} \times 27 = \frac{27\sqrt{3}}{4}$. As a check, the area also equals three triangles each with two sides of 3 and an included angle of 120°: $3 \times \frac{1}{2} \times 9 \times \sin 120^\circ = \frac{27\sqrt{3}}{4}$. Both methods agree.

Applications of de Moivre and roots of unity

Roots of unity appear whenever something repeats around a circle. Engineers use them in the discrete Fourier transform, which breaks a sound, picture or radio signal into pure frequencies and is run billions of times a day inside phones and computers. The three-phase electricity supply that powers factories uses three voltages whose phasors are the cube roots of unity, spaced 120° apart, and this is why the total current in the neutral wire is zero when the load is balanced. In chemistry and crystallography, the symmetry of a molecule with n-fold rotational symmetry is

described by multiplication by an nth root of unity. In every case the underlying idea is the same: multiplying by a number of modulus 1 is a rotation, and n equal rotations bring you back to the start.

Method: revision plan for this chapter

1. Learn the statements of de Moivre's theorem and Euler's formula until you can write them from memory.
2. Practise converting between Cartesian and modulus-argument form quickly and accurately.
3. Derive $\cos 3\theta$ and $\sin 3\theta$ by hand, then attempt $\cos 4\theta$ and $\sin 5\theta$.
4. Practise the expansion of $\left(z \pm \frac{1}{z}\right)^n$ for $n = 2, 3, 4, 5$.
5. Solve at least five equations of the form $z^n = w$ and sketch the roots every time.

Thinking about why de Moivre's theorem is true

It is helpful to understand the reason behind the theorem, not just to apply it. When you multiply two complex numbers in modulus-argument form, the moduli multiply and the arguments add. That fact is a consequence of the addition formulae for sine and cosine. If you multiply a number of modulus 1 by itself, the argument doubles; multiply again and it triples; and so on. After n multiplications the argument has been multiplied by n. Therefore the statement of de Moivre's theorem for a positive whole number n is a simple induction on the product rule. The theorem remains true for negative integers and, with care, for fractions, which is exactly what makes it possible to find roots. In the fractional case there are several possible answers, and that is the reason why the equation $z^n = w$ has n solutions instead of just one.

Worked example 9: using the sum of roots

The roots of $z^5 = 32$ are z_0, z_1, z_2, z_3, z_4. Without calculating them, explain why their sum is zero, and use the result to find the sum of the real parts.

The equation can be written as $z^5 - 32 = 0$, a polynomial with no term in z^4. The sum of the roots of a polynomial equals minus the coefficient of the second highest power, divided by the leading coefficient, so the sum is zero. Because the roots are $2e^{2\pi i k/5}$, we can also see this geometrically: they form a regular pentagon centred at the origin. Taking real parts of the sum, $2\left(1 + \cos 72^\circ + \cos 144^\circ + \cos 216^\circ + \cos 288^\circ\right) = 0$, which gives $\cos 72^\circ + \cos 144^\circ = -\frac{1}{2}$ after using symmetry. This is a quick way to obtain a relation between two trigonometric values without any calculator.

Practical advice for examinations

In examination questions that ask you to show a result, always begin by stating the theorem you are using. Examiners award a mark for writing de Moivre's theorem correctly, another for the expansion, and another for comparing real and imaginary parts. When you find roots, give the arguments in the principal range, between minus

pi and pi, unless the question asks otherwise, and list the roots in order of increasing argument so that the pattern is easy to check. If you sketch the roots, draw the circle on which they lie, label the modulus, and mark the angle between neighbouring roots. A tidy diagram can earn marks and it also helps you notice mistakes, such as a missing root or a root in the wrong quadrant.

Watch out: forgetting the real and imaginary parts

A very common error in multiple-angle questions is to equate the whole expansion with the whole of the right-hand side, forgetting that two complex numbers are equal only if both the real parts and the imaginary parts match. Always say clearly which part you are taking. A second error is to simplify an even power of the wrong function: when you are finding $\cos 5\theta$, only even powers of $\sin\theta$ appear in the real part, so you can replace them using $\sin^2\theta = 1 - \cos^2\theta$. In the imaginary part, only odd powers of $\sin\theta$ appear together with even powers of $\cos\theta$, and you substitute for those instead.

Did you know?

Abraham de Moivre was a French Protestant who fled to London in 1685 to escape religious persecution. He earned his living by tutoring and by solving problems about gambling and life insurance, and he was a friend of Isaac Newton, who is said to have told people with mathematical questions to go to Mr de Moivre because he knew these things better than Newton did. The formula we now call by his name was developed in the 1720s and 1730s. It was Leonhard Euler who later linked it to the exponential function, producing the beautiful equation $e^{i\pi} + 1 = 0$, which joins the five most important numbers in mathematics.

Otto's tip

For $z^n = w$, find one root, then rotate it repeatedly by $\frac{2\pi}{n}$. This is quicker, and the last rotation returning you to the start is a built-in check.