Worksheet - De Moivre's theorem and roots of unity

Year: Further Maths · 20

Show your working.

Question 1

Question 1 – Powers with de Moivre’s theorem *(Section A · Skills practice)*

Use de Moivre’s theorem, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, to evaluate each expression. Give exact answers in the form $a + bi$.

a. $\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^5$
b. $(1+i)^8$
c. $(\sqrt{3} - i)^6$
d. $(-1+i)^5$

Question 2

Question 2 – Euler’s form *(Section A · Skills practice)*

Euler’s formula states that $e^{i\theta} = \cos\theta + i\sin\theta$.

a. Write $3e^{i\pi/3}$ in the form $a + bi$.
b. Write $1 - i$ in the form $re^{i\theta}$.
c. Simplify $2e^{i\pi/4} \times 3e^{i\pi/6}$ and $\dfrac{6e^{i\pi/2}}{2e^{i\pi/3}}$.
d. Show that $\left(e^{i\pi/6}\right)^{12} = 1$ and that $e^{i\pi} + 1 = 0$.

Question 3

Question 3 – Multiple angle formulae *(Section A · Skills practice)*

Let $c = \cos\theta$ and $s = \sin\theta$.

a. Expand $(c + is)^3$ using the binomial theorem.
b. By equating real parts, show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.
c. By equating imaginary parts, show that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.
d. Check both results when $\theta = \frac{\pi}{6}$.

Question 4

Question 4 – Powers of sine and cosine *(Section A · Skills practice)*

Let $z = \cos\theta + i\sin\theta$, so that $z + \dfrac{1}{z} = 2\cos\theta$ and $z - \dfrac{1}{z} = 2i\sin\theta$, and $z^n + \dfrac{1}{z^n} = 2\cos n\theta$.

a. Expand $\left(z + \dfrac{1}{z}\right)^3$ and hence show that $\cos^3\theta = \frac{1}{4}(\cos 3\theta + 3\cos\theta)$.

b. Expand $\left(z - \dfrac{1}{z}\right)^4$ and hence show that $\sin^4\theta = \frac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3)$.

c. Use part (b) to evaluate $\displaystyle\int_0^{\pi/2} \sin^4\theta\, d\theta$.

Question 5

Question 5 – Cube roots *(Section A · Skills practice)*

Solve each equation. Give the roots in Cartesian form with exact values and show them on an Argand diagram.

a. $z^3 = 8$
b. $z^3 = -8i$
c. State the sum of the three roots in each case.

Question 6

Question 6 – Roots of unity *(Section A · Skills practice)*

The sixth roots of unity are the solutions of $z^6 = 1$. Let $\omega = e^{i\pi/3}$.

a. Write down all six roots in the form ω^k and in Cartesian form.
b. Show that $\omega^2 = \omega - 1$ and $\omega^3 = -1$.
c. Show that the sum of the six roots is zero.
d. Find the product of the six roots.

Question 7

Question 7 – The rotating gear wheel *(Section B · Real-life practice)*

A marker on the rim of a gear wheel of radius 5 cm is at the point $z = 5e^{3it}$ in an Argand diagram whose origin is the centre of the wheel. Here t is the time in seconds and the angle is in radians.

a. Find the position of the marker when $t = 2$, in the form $a + bi$ to 2 decimal places.
b. How long does the wheel take to make one complete turn?

c. The velocity of the marker is $\frac{dz}{dt}$. Find it and show that the speed of the marker is constant. Find that speed.
d. Show that the velocity is always perpendicular to the position, by writing $\frac{dz}{dt} = 3i\,z$.

Question 8

Question 8 – The pentagon fountain *(Section B · Real-life practice)*

A garden designer places five sprinklers at the corners of a regular pentagon centred on the fountain. In an Argand diagram with the fountain at the origin and distances in metres, the sprinklers are at the solutions of $z^5 = 1024$.

a. Find the modulus of each sprinkler position and write down all five positions in modulus-argument form.
b. Calculate the distance between two neighbouring sprinklers to 2 decimal places.
c. A path runs along all five sides. Find its total length.
d. Find the area enclosed by the path, using five triangles with their apex at the fountain.

Question 9

Question 9 – The honeycomb pattern *(Section B · Real-life practice)*

A tile designer draws a regular hexagon whose six corners are the roots of $z^6 = -64$ in an Argand diagram, with units in decimetres.

a. Find the modulus and argument of each root.
b. Write the roots in the form $a + bi$.
c. What is the length of one side of the hexagon?
d. Hence find the area of one hexagonal tile to 3 significant figures.

Question 10

Question 10 – Light through a polariser *(Section B · Real-life practice)*

The intensity of light passing through a polarising filter varies as $I = 100\cos^4 \omega t$ watts per square metre, where ω is a constant and t is time.

a. Use $\left(z + \frac{1}{z}\right)^4$ with $z = \cos\omega t + i\sin\omega t$ to show that $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$.
b. Hence write I as a sum of cosines of multiples of ωt.
c. Find the mean intensity over one period.

Question 11

Question 11 – A crystal lattice *(Section B · Real-life practice)*

Four atoms in a square crystal cell are located at the solutions of $z^4 = -16$, where distances are in nanometres.

a. Find the four positions in the form $a + bi$.
b. Show that the four atoms form a square and find its side length.
c. A fifth atom sits at the origin. Find the distance from it to each of the others.

Question 12

Question 12 – The seven-sided coin *(Section B · Real-life practice)*

The British 50p coin is a regular heptagon (seven sides). Model a coin by placing its vertices at the seven solutions of $z^7 = 14^7$, with distances in millimetres.

a. Find the modulus and write the seven vertices in exponential form.
b. Show that the vertices have sum zero.
c. Find the length of one side to 2 decimal places.
d. Find the length of the longest diagonal joining two vertices, to 2 decimal places.

Question 13

Question 13 – Distortion in an amplifier *(Section B · Real-life practice)*

A faulty amplifier turns an input signal $x = 2\sin\theta$ volts into the output $y = x^3$, where $\theta = \omega t$. Sound engineers call the extra frequencies in the output distortion.

a. Use $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ to show that $\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$.
b. Write the output y as a sum of sines of θ and 3θ.
c. Find the amplitude of the third harmonic $\sin 3\theta$ as a percentage of the amplitude of the fundamental $\sin\theta$.

Question 14

Question 14 – The big wheel *(Section B · Real-life practice)*

A giant observation wheel has radius 60 m and its centre is 67.5 m above the ground. It has 32 identical capsules equally spaced round the rim. Let the centre be the origin of an Argand diagram, and let capsule k be at $60e^{i\phi_k}$ where $\phi_k = -\frac{\pi}{2} + \frac{2\pi k}{32}$ at a certain moment, for $k = 0, 1, \ldots, 31$.

a. Show that capsule 0 is at the bottom of the wheel and find its height above the ground.
b. Find the argument of the capsule at the top and its height above the ground.
c. Find the height of capsule 8.
d. Find the sum of the 32 complex positions, and explain the result.

Question 15

Question 15 – An exact value for cos 18 degrees *(Section C · Problem solving)*

In this problem you find an exact value of $\cos\frac{\pi}{10}$.

a. Expand $(\cos\theta + i\sin\theta)^5$ and show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.
b. Explain why $\cos 5\theta = 0$ when $\theta = \frac{\pi}{10}$ and when $\theta = \frac{3\pi}{10}$.
c. Deduce that $\cos^2\frac{\pi}{10}$ is a root of $16x^2 - 20x + 5 = 0$.
d. Hence find the exact value of $\cos^2\frac{\pi}{10}$, and show that $\cos\frac{\pi}{10} = \sqrt{\frac{5+\sqrt{5}}{8}}$.

Question 16

Question 16 – Fourth roots and a square *(Section C · Problem solving)*

Let $w = -8 + 8\sqrt{3}\,i$.

a. Express w in modulus-argument form.
b. Find the four fourth roots of w in the form $r(\cos\theta + i\sin\theta)$ and then in the form $a + bi$.
c. Show the roots on an Argand diagram and prove that they form a square.
d. Find the area of the square.

Question 17

Question 17 – Summing cosines *(Section C · Problem solving)*

Let $C = 1 + \cos\theta + \cos 2\theta + \cdots + \cos(n-1)\theta$ and $S = \sin\theta + \sin 2\theta + \cdots + \sin(n-1)\theta$, and let $z = \cos\theta + i\sin\theta$.

a. Explain why $C + iS = 1 + z + z^2 + \cdots + z^{n-1}$.
b. Sum the geometric series and show that $C + iS = \dfrac{1 - z^n}{1 - z}$.
c. By writing $1 - z^n = -2i\sin\frac{n\theta}{2}\,e^{in\theta/2}$ and similarly for $1 - z$, show that $C = \dfrac{\sin\frac{n\theta}{2}\cos\frac{(n-1)\theta}{2}}{\sin\frac{\theta}{2}}$.
d. Use the formula to find C when $\theta = \frac{\pi}{4}$ and $n = 7$, and check by adding the seven terms.

Question 18

Question 18 – Exact value of cos 72 degrees *(Section C · Problem solving)*

Let $\omega = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$ be a fifth root of unity.

a. Explain why $1+\omega+\omega^2+\omega^3+\omega^4=0$.
b. Let $a=\omega+\omega^4$ and $b=\omega^2+\omega^3$. Show that $a=2\cos 72^\circ$ and $b=2\cos 144^\circ$, and that $a+b=-1$.
c. Show that $ab=-1$.
d. Hence find the exact value of $\cos 72^\circ$.

Question 19

Question 19 – Factorising z to the sixth minus one *(Section C · Problem solving)*

Let $f(z)=z^6-1$.

a. Write down the six roots of $f(z)=0$ in modulus-argument form.
b. Pair each non-real root with its conjugate and show that $f(z)=(z-1)(z+1)(z^2-z+1)(z^2+z+1)$.
c. Show that $z^2+z+1=0$ has the roots $e^{\pm 2\pi i/3}$ and check your factorisation by expanding.
d. Hence prove that $\cos\frac{\pi}{3}=\frac{1}{2}$ using the sum of two conjugate roots.

Question 20

Question 20 – Challenge: a polynomial with a pattern of roots *(Section D · Challenge)*

Consider the equation $(z+1)^5=z^5$.

a. Explain why $z=0$ is not a root and why the equation reduces to a polynomial of degree 4.
b. Put $w=\dfrac{z+1}{z}$. Show that $w^5=1$ and $w\neq 1$.
c. Hence show that $z=\dfrac{1}{w-1}$ and find z in terms of $\theta=\frac{2\pi k}{5}$ for $k=1,2,3,4$.
d. Show that $\dfrac{1}{e^{i\theta}-1}=-\dfrac{1}{2}-\dfrac{i}{2}\cot\frac{\theta}{2}$. Hence write all four roots and check one numerically.