Answers - De Moivre's theorem and roots of unity

Year: Further Maths

Answer to Question 1

Method: write the number in modulus-argument form $r(\cos\theta + i\sin\theta)$, then use de Moivre: $[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$. Reduce the angle $n\theta$ by multiples of 2π before evaluating.

a. The number is already in modulus-argument form with $r = 1$, so multiply the angle by 5. $\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$ This point is on the unit circle in the second quadrant.

b. $|1+i| = \sqrt{2}$ and $\arg(1+i) = \frac{\pi}{4}$; the angle becomes $8 \times \frac{\pi}{4} = 2\pi$.
$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$, so $(1+i)^8 = (\sqrt{2})^8(\cos 2\pi + i\sin 2\pi) = 16 \times 1 =$ **16**
So $(1+i)^8 = 16$, a real number.

c. $|\sqrt{3} - i| = 2$ and the point is in the fourth quadrant.
$\sqrt{3} - i = 2\left(\cos(-\frac{\pi}{6}) + i\sin(-\frac{\pi}{6})\right)$, so the sixth power is
$2^6(\cos(-\pi) + i\sin(-\pi)) = 64 \times (-1) = -64$ So $(\sqrt{3} - i)^6 = -64$.

d. $|-1+i| = \sqrt{2}$ and the point is in the second quadrant; $(\sqrt{2})^5 = 4\sqrt{2}$.
$-1 + i = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$. The fifth power is
$4\sqrt{2}\left(\cos\frac{15\pi}{4} + i\sin\frac{15\pi}{4}\right) = 4\sqrt{2}\left(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4})\right) = 4\sqrt{2} \cdot \frac{\sqrt{2}}{2}(1-i) =$
$4 - 4i$ Since $\frac{15\pi}{4} - 4\pi = -\frac{\pi}{4}$, the result is $4 - 4i$.

Answer to Question 2

Method: in exponential form $re^{i\theta}$, the modulus is r and the argument is θ. Multiplying multiplies the moduli and adds the exponents, exactly like the laws of indices.

a. Use $\cos\frac{\pi}{3} = \frac{1}{2}$ and $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$. $3\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = \frac{3}{2} + \frac{3\sqrt{3}}{2}i$

b. $1 - i$ is in the fourth quadrant with reference angle $\frac{\pi}{4}$. $|1-i| = \sqrt{2}$, $\arg = -\frac{\pi}{4}$, so $\sqrt{2}\,e^{-i\pi/4}$

c. Multiply moduli and add exponents: $6e^{i(\pi/4+\pi/6)} = 6e^{5i\pi/12}$. Divide: $3e^{i(\pi/2-\pi/3)} = 3e^{i\pi/6}$.

d. Use the laws of indices, then Euler's formula. $\left(e^{i\pi/6}\right)^{12} = e^{2\pi i} = \cos 2\pi + i\sin 2\pi =$ **1**. Also $e^{i\pi} = \cos\pi + i\sin\pi = -1$, so $e^{i\pi} + 1 =$ **0**. The second result links the five most important numbers 0, 1, e, i and π.

Answer to Question 3

Method: by de Moivre, $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$. Expanding the left-hand side with the binomial theorem and comparing real and imaginary parts gives multiple angle formulae; use $s^2 = 1 - c^2$ or $c^2 = 1 - s^2$ to keep one function.

a. $(c+is)^3 = c^3 + 3c^2(is) + 3c(is)^2 + (is)^3 = c^3 + 3ic^2s - 3cs^2 - is^3$
b. Real part: $c^3 - 3cs^2 = c^3 - 3c(1-c^2) = 4c^3 - 3c$. By de Moivre the real part is also $\cos 3\theta$.
c. Imaginary part: $3c^2s - s^3 = 3(1-s^2)s - s^3 = 3s - 4s^3$, which equals $\sin 3\theta$.
d. $\theta = \frac{\pi}{6}$: $\cos\frac{\pi}{2} = 0$ and $4\left(\frac{\sqrt{3}}{2}\right)^3 - 3\cdot\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2} - \frac{3\sqrt{3}}{2} = 0$. $\sin\frac{\pi}{2} = 1$ and $3\cdot\frac{1}{2} - 4\cdot\frac{1}{8} = 1$. Both correct.

Answer to Question 4

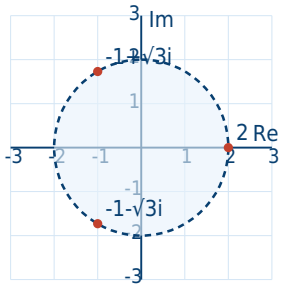
Method: with $z = \cos\theta + i\sin\theta$, use $z^n + \dfrac{1}{z^n} = 2\cos n\theta$ and $z^n - \dfrac{1}{z^n} = 2i\sin n\theta$.

Expand with the binomial theorem, pair the terms z^k and z^{-k}, and compare with the power of $2\cos\theta$ or $2i\sin\theta$.

a. Use the binomial coefficients 1, 3, 3, 1 and pair z^3 with $\frac{1}{z^3}$ and $3z$ with $\frac{3}{z}$. $\left(z+\frac{1}{z}\right)^3 = z^3 + 3z + \frac{3}{z} + \frac{1}{z^3} = \left(z^3 + \frac{1}{z^3}\right) + 3\left(z + \frac{1}{z}\right) = 2\cos 3\theta + 6\cos\theta$. The left side is $8\cos^3\theta$. So $\cos^3\theta = \frac{1}{4}(\cos 3\theta + 3\cos\theta)$. Divide both sides by 8 to obtain the result.
b. Use the coefficients 1, 4, 6, 4, 1 with alternating signs; $(2i)^4 = 16$. $\left(z-\frac{1}{z}\right)^4 = z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4} = 2\cos 4\theta - 8\cos 2\theta + 6$. The left side is $(2i\sin\theta)^4 = 16\sin^4\theta$. So $\sin^4\theta = \frac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3)$. This formula lets us integrate $\sin^4\theta$, which cannot be done directly.
c. Integrate each cosine term; at $\frac{\pi}{2}$ the sine terms are $\sin 2\pi = 0$ and $\sin\pi = 0$. $\frac{1}{8}\left[\frac{\sin 4\theta}{4} - 2\sin 2\theta + 3\theta\right]_0^{\pi/2} = \frac{1}{8}\left(0 - 0 + \frac{3\pi}{2}\right) = \dfrac{3\pi}{16}$ So the integral equals $\frac{3\pi}{16} \approx 0.589$.

Answer to Question 5

a. $8 = 8(\cos 0 + i\sin 0)$, so $z = 2\left(\cos\frac{2\pi k}{3} + i\sin\frac{2\pi k}{3}\right)$ for $k = 0, 1, 2$: **2**, $-1 + \sqrt{3}\,i$, $-1 - \sqrt{3}\,i$
b. $-8i = 8\left(\cos(-\frac{\pi}{2}) + i\sin(-\frac{\pi}{2})\right)$, so $z = 2\left(\cos\left(-\frac{\pi}{6} + \frac{2\pi k}{3}\right) + i\sin(\ldots)\right)$: arguments $-\frac{\pi}{6}, \frac{\pi}{2}, \frac{7\pi}{6}$. The roots are $\sqrt{3} - i$, $2i$, $-\sqrt{3} - i$.
c. Each set of roots forms an equilateral triangle centred on the origin, so the sum is **0** in both cases (there is no z^2 term, so the sum of the roots is 0).



Answer to Question 6

a. $\omega^k = \cos\frac{k\pi}{3} + i\sin\frac{k\pi}{3}$: 1, $\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$, -1, $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$, $\frac{1}{2} - \frac{\sqrt{3}}{2}i$

b. $\omega^2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $\omega - 1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. They are equal. $\omega^3 = e^{i\pi} = -1$.

c. Pair each root with its negative: $1 + (-1) = 0$, $\omega + \omega^4 = 0$, $\omega^2 + \omega^5 = 0$. The total is **0**. (The coefficient of z^5 in $z^6 - 1$ is zero.)

d. The product of the roots of $z^6 - 1 = 0$ is $(-1)^6 \times (-1) = -1$. Check: the arguments sum to $\frac{\pi}{3}(0 + 1 + 2 + 3 + 4 + 5) = 5\pi$, giving $\cos 5\pi = -1$.

Answer to Question 7

a. $z = 5(\cos 6 + i\sin 6)$. $\cos 6 \approx 0.9602$ and $\sin 6 \approx -0.2794$. So $z \approx 4.80 - 1.40i$ **cm**.

b. One turn needs $3T = 2\pi$, so $T = \dfrac{2\pi}{3} \approx$ **2.09 s**.

c. $\dfrac{dz}{dt} = 15ie^{3it}$, and $|ie^{3it}| = 1$ so the speed is $|15| =$ **15 cm s^{-1}**, which does not depend on t.

d. Multiplying by i rotates a number through 90°. Since $\dfrac{dz}{dt} = 3iz$ and 3 is a positive real number, the velocity points 90° anticlockwise from the position, which is the direction of the tangent.

Answer to Question 8

a. $|z|^5 = 1024$, so $|z| = 4$. The positions are $4\left(\cos\frac{2\pi k}{5} + i\sin\frac{2\pi k}{5}\right)$ for $k = 0, 1, 2, 3, 4$.

b. The chord subtending an angle $\frac{2\pi}{5}$ at the centre has length $2 \times 4\sin\frac{\pi}{5} = 8\sin 36^\circ \approx$ **4.70 m**.

c. $5 \times 4.70 =$ **23.5 m** (23.51 m to 2 decimal places).

d. Each triangle has area $\frac{1}{2} \times 4 \times 4 \times \sin 72^\circ$. Five triangles: $40\sin 72^\circ \approx$ **38.04 m^2**.

Answer to Question 9

a. $-64 = 64(\cos\pi + i\sin\pi)$, so $|z| = 2$ and $\arg z = \dfrac{\pi + 2\pi k}{6}$: the arguments are $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$.
b. $\sqrt{3}+i$, $2i$, $-\sqrt{3}+i$, $-\sqrt{3}-i$, $-2i$, $\sqrt{3}-i$
c. In a regular hexagon the side equals the distance from the centre to a corner, so the side is **2 dm**. Check: $|2i - (\sqrt{3}+i)| = |-\sqrt{3}+i| = 2$.
d. Six equilateral triangles of side 2: $6 \times \frac{\sqrt{3}}{4} \times 4 = 6\sqrt{3} \approx$ **10.4 dm**^{2}.

Answer to Question 10

Method: with $z = \cos\theta + i\sin\theta$, use $z^n + \dfrac{1}{z^n} = 2\cos n\theta$ and $z^n - \dfrac{1}{z^n} = 2i\sin n\theta$.

Expand with the binomial theorem, pair the terms z^k and z^{-k}, and compare with the power of $2\cos\theta$ or $2i\sin\theta$.

a. $\left(z + \frac{1}{z}\right)^4 = z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4} = 2\cos 4\theta + 8\cos 2\theta + 6$. The left side is $16\cos^4\theta$. So $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$.
b. $I = 100\cos^4\omega t = 12.5\cos 4\omega t + 50\cos 2\omega t + 37.5$
c. Over a full period the mean of $\cos 4\omega t$ and $\cos 2\omega t$ is zero, so the mean is the constant term, **37.5 W m**$^{-2}$.

Answer to Question 11

a. $-16 = 16(\cos\pi + i\sin\pi)$. $|z| = 2$ and $\arg z = \frac{\pi + 2\pi k}{4}$: $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$. Positions $\sqrt{2}+\sqrt{2}\,i$, $-\sqrt{2}+\sqrt{2}\,i$, $-\sqrt{2}-\sqrt{2}\,i$, $\sqrt{2}-\sqrt{2}\,i$.
b. Neighbouring points differ by $2\sqrt{2}$ in one coordinate only, so every side is $2\sqrt{2} \approx 2.83$ **nm** and the angles are right angles. It is a square (the roots are equally spaced round a circle with four of them).
c. Each lies at distance equal to the modulus: **2 nm**.

Answer to Question 12

a. $|z| = 14$, vertices $14e^{2\pi ik/7}$ for $k = 0, \ldots, 6$.
b. The vertices are $14\omega^k$ with $\omega = e^{2\pi i/7}$. Their sum is the geometric series $14 \cdot \dfrac{1-\omega^7}{1-\omega} = 14 \cdot \dfrac{1-1}{1-\omega} =$ **0**.
c. Side $= 2 \times 14\sin\frac{\pi}{7} = 28\sin\frac{\pi}{7} \approx$ **12.15 mm**.
d. The longest diagonal skips two vertices and subtends $3 \times \frac{2\pi}{7}$ at the centre: $28\sin\frac{3\pi}{7} \approx$ **27.30 mm**. (The coin has constant width because each vertex is opposite a side.)

Answer to Question 13

Method: powers of $\sin\theta$ can be written as sums of sines of multiples of θ by rearranging a multiple angle formula. Each term then represents a separate frequency in the signal.

a. Make $\sin^3\theta$ the subject of the triple angle formula. Rearrange: $4\sin^3\theta = 3\sin\theta - \sin 3\theta$, so $\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$.
b. Cube the input: $x^3 = 8\sin^3\theta$, then use part a). $y = 8\sin^3\theta = 2(3\sin\theta - \sin 3\theta) = 6\sin\theta - 2\sin 3\theta$ So the output contains the original frequency and a new frequency three times higher.
c. Compare the two amplitudes. The amplitudes are 6 V and 2 V, so the third harmonic is $\dfrac{2}{6} = \dfrac{1}{3} \approx$ **33.3 %** of the fundamental. So the distortion is large: one third of the size of the main signal.

Answer to Question 14

a. $\phi_0 = -\frac{\pi}{2}$ so $z = 60e^{-i\pi/2} = -60i$, directly below the centre. Height $= 67.5 - 60 =$ **7.5 m**.
b. The top has argument $\frac{\pi}{2}$, so $-\frac{\pi}{2} + \frac{2\pi k}{32} = \frac{\pi}{2}$ gives $k = 16$. Height $= 67.5 + 60 =$ **127.5 m**.
c. $\phi_8 = -\frac{\pi}{2} + \frac{\pi}{2} = 0$, so the capsule is level with the centre: **67.5 m**.
d. The positions are $-60i \times \omega^k$ with $\omega = e^{2\pi i/32}$, a geometric series with ratio $\omega \neq 1$ and $\omega^{32} = 1$, so the sum is **0**. By symmetry the capsules balance about the centre.

Answer to Question 15

a. The real part of $(c + is)^5$ is $c^5 - 10c^3s^2 + 5cs^4$. Put $s^2 = 1 - c^2$: $c^5 - 10c^3 + 10c^5 + 5c(1 - 2c^2 + c^4) = c^5 - 10c^3 + 10c^5 + 5c - 10c^3 + 5c^5 = 16c^5 - 20c^3 + 5c$.
b. $5\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$ respectively, and $\cos\frac{\pi}{2} = \cos\frac{3\pi}{2} = 0$.
c. $c(16c^4 - 20c^2 + 5) = 0$ with $c \neq 0$ for these angles, so $16c^4 - 20c^2 + 5 = 0$; put $x = c^2$ to get $16x^2 - 20x + 5 = 0$.
d. $x = \dfrac{20 \pm \sqrt{400 - 320}}{32} = \dfrac{5 \pm \sqrt{5}}{8}$. Since $\frac{\pi}{10}$ is smaller than $\frac{3\pi}{10}$, its cosine is larger, so we take the larger root: $\cos^2\frac{\pi}{10} = \dfrac{5 + \sqrt{5}}{8} \approx 0.9045$,

$\cos\frac{\pi}{10} = \sqrt{\dfrac{5 + \sqrt{5}}{8}} \approx 0.9511.$

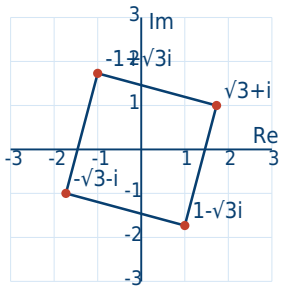
Answer to Question 16

a. $|w| = \sqrt{64 + 192} = 16$ and $\arg w = \pi - \arctan\sqrt{3} = \frac{2\pi}{3}$: $16\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$.

b. $z = 2\left(\cos\left(\frac{\pi}{6} + \frac{k\pi}{2}\right) + i\sin(\ldots)\right)$: arguments $\frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$. In Cartesian form: $\sqrt{3} + i$, $-1 + \sqrt{3}\,i$, $-\sqrt{3} - i$, $1 - \sqrt{3}\,i$.

c. The roots have equal modulus 2 and consecutive arguments differ by $\frac{\pi}{2}$. A quadrilateral inscribed in a circle with equal angles of 90° at the centre is a square (diagonals are equal diameters that cross at right angles and bisect each other).

d. The diagonal is a diameter of length 4, so the area is $\frac{1}{2} \times 4 \times 4 = \mathbf{8}$. (Side $2\sqrt{2}$, area 8.)



Answer to Question 17

a. By de Moivre, $z^k = \cos k\theta + i\sin k\theta$. Adding for $k = 0, 1, \ldots, n-1$ gives the real part C and imaginary part S.

b. The series is geometric with first term 1 and common ratio z, so the sum is $\dfrac{1 - z^n}{1 - z}$ (valid for $z \neq 1$).

c. $1 - e^{in\theta} = e^{in\theta/2}\left(e^{-in\theta/2} - e^{in\theta/2}\right) = -2i\sin\frac{n\theta}{2}\,e^{in\theta/2}$. Likewise $1 - z = -2i\sin\frac{\theta}{2}\,e^{i\theta/2}$. The quotient is $\dfrac{\sin(n\theta/2)}{\sin(\theta/2)}e^{i(n-1)\theta/2}$. The real part is

$$C = \frac{\sin\frac{n\theta}{2}\cos\frac{(n-1)\theta}{2}}{\sin\frac{\theta}{2}}.$$

d. $C = \dfrac{\sin\frac{7\pi}{8}\cos\frac{3\pi}{4}}{\sin\frac{\pi}{8}} = \dfrac{\sin\frac{\pi}{8}\cdot\left(-\frac{\sqrt{2}}{2}\right)}{\sin\frac{\pi}{8}} = -\dfrac{\sqrt{2}}{2} \approx -0.707$. Check: the terms are $1, \frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}, -1, -\frac{\sqrt{2}}{2}, 0$, with sum $-\frac{\sqrt{2}}{2}$.

Answer to Question 18

a. The numbers $1, \omega, \ldots, \omega^4$ are the roots of $z^5 - 1 = 0$, which has no z^4 term, so

their sum is **0**. (Or sum the geometric series: $\dfrac{1-\omega^5}{1-\omega}=0$.)

b. because $\omega^5=1$, so $\omega+\omega^4=2\operatorname{Re}\omega=2\cos 72^\circ$. Likewise and $b=2\cos 144^\circ$. From part (a), $a+b=-1$.

c. $ab=\omega^3+\omega^4+\omega^6+\omega^7=\omega^3+\omega^4+\omega+\omega^2=-1$.

d. a and b are the roots of $t^2+t-1=0$, so $t=\dfrac{-1\pm\sqrt{5}}{2}$. Since $a=2\cos 72^\circ>0$, $a=\dfrac{\sqrt{5}-1}{2}$, so $\cos 72^\circ=\dfrac{\sqrt{5}-1}{4}\approx 0.309$.

Answer to Question 19

a. $z=\cos\frac{k\pi}{3}+i\sin\frac{k\pi}{3}$ for $k=0,1,2,3,4,5$.

b. The roots are $1,-1,e^{\pm i\pi/3},e^{\pm 2i\pi/3}$. A conjugate pair $e^{\pm i\alpha}$ gives the real quadratic $z^2-2\cos\alpha\, z+1$. For $\alpha=\frac{\pi}{3}$: z^2-z+1 (using $2\cos\frac{\pi}{3}=1$). For $\alpha=\frac{2\pi}{3}$: z^2+z+1 (using $2\cos\frac{2\pi}{3}=-1$). So $f(z)=(z-1)(z+1)(z^2-z+1)(z^2+z+1)$.

c. $z=\dfrac{-1\pm\sqrt{-3}}{2}=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i=e^{\pm 2\pi i/3}$. Expansion: $(z^2-z+1)(z^2+z+1)=z^4+z^2+1$ and $(z-1)(z+1)=z^2-1$. Then $(z^2-1)(z^4+z^2+1)=z^6-1$. Correct.

d. The roots of $z^2-z+1=0$ are $\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$. Their sum is $1=2\cos\frac{\pi}{3}$, so $\cos\frac{\pi}{3}=\frac{1}{2}$.

Answer to Question 20

a. $z=0$ gives $1=0$, false. Expanding, the z^5 terms cancel, leaving $5z^4+10z^3+10z^2+5z+1=0$, degree 4.

b. Divide by z^5 (valid as $z\neq 0$): $\left(\dfrac{z+1}{z}\right)^5=1$. Also $w=1+\frac{1}{z}\neq 1$.

c. $w-1=\dfrac{1}{z}$, so $z=\dfrac{1}{w-1}$. The fifth roots of unity other than 1 are $w=e^{i\theta}$, $\theta=\frac{2\pi k}{5}$, $k=1,2,3,4$.

d. $e^{i\theta}-1=e^{i\theta/2}\cdot 2i\sin\frac{\theta}{2}$, so $\dfrac{1}{e^{i\theta}-1}=\dfrac{e^{-i\theta/2}}{2i\sin(\theta/2)}=\dfrac{\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}}{2i\sin\frac{\theta}{2}}=-\frac{1}{2}-\frac{i}{2}\cot\frac{\theta}{2}$. With $\frac{\theta}{2}=\frac{k\pi}{5}$ the roots are $z=-\frac{1}{2}-\frac{i}{2}\cot\frac{k\pi}{5}$, $k=1,2,3,4$. (Equivalently $-\frac{1}{2}+\frac{i}{2}\cot\frac{k\pi}{5}$ because $\cot\frac{(5-k)\pi}{5}=-\cot\frac{k\pi}{5}$.) Check $k=1$: $\cot 36^\circ\approx 1.3764$, $z\approx -0.5-0.688i$, and $(z+1)^5=(0.5-0.688i)^5$ and z^5 both have modulus $0.8507^5\approx 0.4473$ with matching argument: verified numerically.