Test answers - Complex numbers

Year: Further Maths · Duration: 60 minutes · /20

Answer to Question 1 (3 marks)

Method: treat i like a letter and replace i^2 by -1. To divide, multiply the top and bottom by the conjugate of the denominator, so that the denominator becomes the real number $a^2 + b^2$. Mark scheme as shown in brackets.

a. The conjugate of z_2 is $1 + i$ and $(1 - i)(1 + i) = 2$.
$\dfrac{(2 + 3i)(1 + i)}{(1 - i)(1 + i)} = \dfrac{2 + 2i + 3i - 3}{2} = \dfrac{-1 + 5i}{2}$. [1 for conjugate, 1 for result]
$-\frac{1}{2} + \frac{5}{2}i$

b. Use $|z_1 z_2| = |z_1||z_2|$, which avoids multiplying out. $|z_1||z_2| = \sqrt{13} \times \sqrt{2} = \sqrt{26}$ [1]
Check: $z_1 z_2 = 5 + i$ and $|5 + i| = \sqrt{26}$.

Answer to Question 2 (4 marks)

Method: a polynomial with real coefficients has its non-real roots in conjugate pairs $p \pm qi$. For a quadratic, use the formula with $\sqrt{-k} = i\sqrt{k}$, or use the sum and product of the roots. Mark scheme as shown in brackets.

a. Substitute $z = 1$. $f(1) = 1 - 3 + 7 - 5 = 0$. [1] So $(z - 1)$ is a factor by the factor theorem.

b. Divide by $(z - 1)$ and solve the quadratic factor with the formula.
$f(z) = (z - 1)(z^2 - 2z + 5)$ [1]. $z = \dfrac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i$ (and 1) [1] So the roots are 1, $1 + 2i$ and $1 - 2i$.

c. Points $(1, 0)$, $(1, 2)$, $(1, -2)$ on the line $x = 1$. [1] The two non-real roots are reflections of each other in the real axis.

Answer to Question 3 (4 marks)

Method: for $z = x + iy$, $|z| = \sqrt{x^2 + y^2}$. Plot the point first to find the quadrant, then find the reference angle and adjust it so that $-\pi < \arg z \leq \pi$. Mark scheme as shown in brackets.

a. u is in the second quadrant with reference angle $\frac{\pi}{3}$; v is in the fourth quadrant with reference angle $\frac{\pi}{4}$. $|u| = \sqrt{9 + 27} = 6$, $\arg u = \frac{2\pi}{3}$ [1]; $|v| = 2\sqrt{2}$, $\arg v = -\frac{\pi}{4}$ [1]

b. Divide the moduli and subtract the arguments; $\frac{11\pi}{12}$ lies in the required range.

■ $\frac{3\sqrt{2}}{2}$ [1]; $\arg\frac{u}{v} = \frac{2\pi}{3} + \frac{\pi}{4} = \frac{11\pi}{12}$ [1]

Answer to Question 4 (6 marks)

Method: $|z-a|$ is the distance from z to the point a. Points equidistant from two points lie on the perpendicular bisector; substitute $z = x + iy$ and square to get Cartesian equations. Mark scheme as shown in brackets.

a. $|z-2| = |z-6i|$ [1]. $(x-2)^2 + y^2 = x^2 + (y-6)^2$ [1] gives $-4x + 4 = -12y + 36$, so $x - 3y = -8$ [1]
b. $(x-2)^2 + y^2 = 25$ with $x = 3y - 8$: $(3y-10)^2 + y^2 = 25$ [1], $10y^2 - 60y + 75 = 0$, $2y^2 - 12y + 15 = 0$ [1], $y = \frac{6 \pm \sqrt{6}}{2}$, $x = 3y - 8 = \frac{2 \pm 3\sqrt{6}}{2}$. Points $\left(\frac{2+3\sqrt{6}}{2}, \frac{6+\sqrt{6}}{2}\right)$ and $\left(\frac{2-3\sqrt{6}}{2}, \frac{6-\sqrt{6}}{2}\right)$ [1]

Check: the midpoint of AB, $(1, 3)$, satisfies $1 - 9 = -8$, as it should.

Answer to Question 5 (3 marks)

Method: a polynomial with real coefficients has its non-real roots in conjugate pairs $p \pm qi$. For a quadratic, use the formula with $\sqrt{-k} = i\sqrt{k}$, or use the sum and product of the roots. Mark scheme as shown in brackets.

a. $2 + i$ (conjugate) [1]
b. Sum $= 4 = -p$, so $p = -4$ [1]. Product $= (2-i)(2+i) = 5 = q$, so $q = 5$ [1]

So the equation is $z^2 - 4z + 5 = 0$.

Check: $(2-i)^2 - 4(2-i) + 5 = 3 - 4i - 8 + 4i + 5 = 0$ ✓.