Lesson - Complex numbers

Year: Further Maths

Complex numbers extend the real numbers so that every quadratic, and in fact every polynomial, has roots. This chapter covers Cartesian form, modulus-argument form, polynomial roots and loci on the Argand diagram.

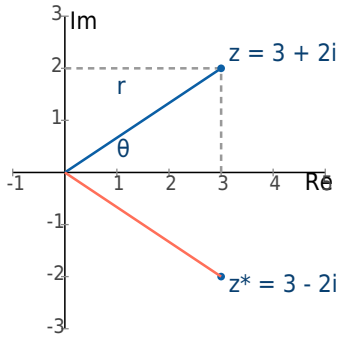
Starter activity

A GPS app places your position at $3 + 2i$ on a map in kilometres. Walk 4 km east and 1 km north: what is the new complex number? Now turn the whole map through 90° anticlockwise about the origin. Where does your original position go? Try multiplying $3 + 2i$ by i and see what you get. Compare the result with the rotated point. What do you think multiplying by a complex number does to a position on the map?

Cartesian form and the Argand diagram

Definition: complex number

A complex number is $z = x + iy$ with x, y real and $i^2 = -1$. We write $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$. On an **Argand diagram** the number is the point (x, y), with the real axis horizontal and the imaginary axis vertical.



Argand diagram: z, its conjugate, modulus r and argument θ.

Definition: conjugate, modulus and argument

The **conjugate** is $z^* = x - iy$. The **modulus** is $|z| = r = \sqrt{x^2 + y^2}$. The **argument** $\theta = \arg z$ is the anticlockwise angle from the positive real axis, taken in the principal range $-\pi$.

Key fact: operations in Cartesian form

- Add or subtract the real and imaginary parts separately.
- $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$.

- $zz^* = x^2 + y^2 = |z|^2$, which is always real.

Method: dividing complex numbers

1. Write the quotient as a fraction.
2. Multiply the numerator and denominator by the conjugate of the denominator.
3. Expand both, using $i^2 = -1$; the denominator becomes the real number $|z|^2$.
4. Split into real and imaginary parts.

Example 1

Write $\dfrac{5+i}{2-3i}$ in the form $a+bi$.

$$\frac{(5+i)(2+3i)}{(2-3i)(2+3i)} = \frac{10+15i+2i-3}{13} = \frac{7+17i}{13}.$$

Quadratics and polynomials

Theorem: conjugate root theorem

If a polynomial has **real coefficients** and z_1 is a root, then z_1^* is also a root. So a real quadratic with $b^2 - 4ac$ has roots $p \pm qi$, a real cubic has 3 real roots or 1 real and a conjugate pair, and a real quartic can have two conjugate pairs.

Method: using a known complex root

1. Write down the conjugate as a second root.
2. Form the real quadratic factor $z^2 - 2\,\mathrm{Re}(z_1)z + |z_1|^2$.
3. Divide the polynomial by this factor, or by $(z - \text{real root})$, to find the remaining roots.
4. Check by substituting one root back.

Example 2

Solve $z^3 - 5z^2 + 17z - 13 = 0$, given that $z = 1$ is a root.

Divide by $(z-1)$: $(z-1)(z^2 - 4z + 13) = 0$. Then $z^2 - 4z + 13 = 0$ gives $z = 2 \pm 3i$. The roots are 1 and $2 \pm 3i$.

Example 3: square root of a complex number

Find $\sqrt{5+12i}$.

Put $(x+iy)^2 = 5 + 12i$. Equating real and imaginary parts gives $x^2 - y^2 = 5$ and

$2xy = 12$. From the second, $y = 6/x$, so $x^4 - 5x^2 - 36 = 0$, $(x^2 - 9)(x^2 + 4) = 0$, and $x = \pm 3$, $y = \pm 2$. So $\sqrt{5 + 12i} = \pm(3 + 2i)$.

Modulus-argument form

Rule: modulus-argument form

$z = r(\cos\theta + i\sin\theta)$. For products and quotients:

$|z_1 z_2| = |z_1||z_2|$ and $\arg(z_1 z_2) = \arg z_1 + \arg z_2$;

and $\arg\dfrac{z_1}{z_2} = \arg z_1 - \arg z_2$,

adjusting by 2π to return to the principal range. Multiplying by z_2 is an enlargement by $|z_2|$ and a rotation through $\arg z_2$.

Method: finding the argument

1. Sketch the point on an Argand diagram and note its quadrant.
2. Find the acute reference angle .
3. Use $\theta = \alpha$, $\pi - \alpha$, $-\pi + \alpha$ or $-\alpha$ for quadrants 1, 2, 3, 4.
4. Write $z = r(\cos\theta + i\sin\theta)$.

Example 4

Write $z = -1 + \sqrt{3}\,i$ in modulus-argument form.

$r = \sqrt{1 + 3} = 2$. The point is in the second quadrant with reference angle $\arctan\sqrt{3} = \frac{\pi}{3}$, so $\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$. Hence $z = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$.

Watch out

$\theta = \arctan(y/x)$ is only correct when $x > 0$. Always sketch the point first, otherwise you will give an argument in the wrong quadrant.

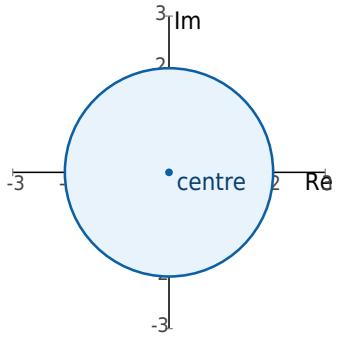
Loci in the Argand diagram

Definition: locus

A **locus** is the set of all points z in the Argand diagram that satisfy a given condition.

Key fact: standard loci

Condition	Locus
$\lvert z-a\rvert = r$	circle, centre a, radius r (inside: $\lvert z-a\rvert \le r$)
$\lvert z-a\rvert = \lvert z-b\rvert$	perpendicular bisector of the line joining a and b
$\arg(z-a) = \theta$	half-line from a (not including a) at angle θ to the positive real direction



The locus |z| = 2 is a circle; |z| ≤ 2 is the shaded disc.

Method: Cartesian equation of a locus

1. Put $z = x + iy$.
2. Write each modulus as a square root and square both sides.
3. Expand and simplify to an equation in x and y.
4. Identify the curve and state any excluded points.

Example 5

Find the Cartesian equation of the locus $\lvert z-4\rvert = \lvert z-2i\rvert$.

With $z = x+iy$: $(x-4)^2 + y^2 = x^2 + (y-2)^2$, so $-8x + 16 = -4y + 4$, giving $y = 2x - 3$. This is the perpendicular bisector of 4 and $2i$; it passes through their midpoint $(2, 1)$ because $1 = 2 \times 2 - 3$.

Example 6

Solve $\dfrac{1}{z} + \dfrac{1}{1+i} = 1$.

The second fraction is $\dfrac{1-i}{2}$, so $\dfrac{1}{z} = 1 - \dfrac{1-i}{2} = \dfrac{1+i}{2}$. Then $z = \dfrac{2}{1+i} = \dfrac{2(1-i)}{2} = 1 - i$.

Note: greatest and least modulus

On the circle $\lvert z-c\rvert = r$, the greatest and least values of $\lvert z\rvert$ are $\lvert c\rvert + r$ and ,

found along the line through the origin and the centre.

Watch out

- Never leave i in a denominator.
- The conjugate root theorem needs *real* coefficients.
- The half-line $\arg(z-a)=\theta$ does not include its starting point a; draw an open circle there.

Worked example 7: a quartic with one known complex root

Given that $z=1+i$ is a root of $z^4-2z^3+4z^2-4z+4=0$, find all the roots.

The coefficients are real, so $1-i$ is also a root. The sum of this pair is 2 and the product is 2, so z^2-2z+2 is a factor. Divide the quartic by it: the quotient has leading term z^2, and matching the constant term $4=2\times 2$ gives z^2+2. Expand to check: $(z^2-2z+2)(z^2+2)=z^4-2z^3+2z^2+2z^2-4z+4=z^4-2z^3+4z^2-4z+4$, which is the original. The second factor gives $z^2=-2$, so $z=\pm i\sqrt{2}$. The four roots are $1\pm i$ and $\pm i\sqrt{2}$.

Method: checking a claimed root first

1. Substitute the claimed root into the polynomial, using $i^2=-1$ at each step.
2. If the result is zero, write down its conjugate as a second root and form the real quadratic factor $z^2-2az+(a^2+b^2)$.
3. Divide to find the quadratic that remains and solve it.
4. Check that the sum and product of all roots match the coefficients.

Worked example 8: powers of i and simplifying

Simplify $\dfrac{(1+i)^2}{1-i}$ and write the answer in the form $a+bi$.

The numerator is $(1+i)^2=1+2i+i^2=2i$. So the fraction is $\dfrac{2i}{1-i}$. Multiply the top and the bottom by the conjugate $1+i$: $\dfrac{2i(1+i)}{(1-i)(1+i)}=\dfrac{2i+2i^2}{2}=\dfrac{2i-2}{2}=-1+i$.

As a check, multiply back: $(-1+i)(1-i)=-1+i+i-i^2=2i$, which equals the numerator.

Worked example 9: the argument of a quotient

Show that $\arg\left(\dfrac{1+i\sqrt{3}}{1-i}\right)=\frac{7\pi}{12}$.

The argument of $1+i\sqrt{3}$ is $\frac{\pi}{3}$, found from the reference angle $\arctan\sqrt{3}$ in the first quadrant. The argument of $1-i$ is $-\frac{\pi}{4}$, because the point is in the fourth quadrant. For a quotient we subtract arguments, so the answer is $\frac{\pi}{3}-\left(-\frac{\pi}{4}\right)=\frac{7\pi}{12}$. This lies in the principal range, so no adjustment by 2π is needed.

Worked example 10: a locus with an inequality

Sketch the region where $|z-2i|\leq 2$ and $\mathrm{Im}(z)\geq 2$.

The first condition is the disc with centre $2i$ and radius 2. The second is everything on or above the horizontal line $y=2$. The line passes through the centre, so the region is the upper half of the disc. Its area is half of $\pi\times 2^2$, which is 2π, and its boundary consists of a semicircle and the diameter from $-2+2i$ to $2+2i$. Remember to say whether the boundary is included: here it is, because both inequalities are not strict.

Note: why multiplication rotates

Write $z=r(\cos\theta+i\sin\theta)$ and multiply by $w=s(\cos\phi+i\sin\phi)$. The product has modulus rs and argument $\theta+\phi$, so multiplying by w stretches the Argand diagram by the factor s and turns it through the angle ϕ. This is why multiplying by i (modulus 1, argument $\frac{\pi}{2}$) is a quarter turn anticlockwise. In engineering the same idea describes a phasor: an alternating voltage is a rotating arrow, and multiplying by a complex impedance both scales it and shifts its phase.

Watch out: common examination errors

- Writing $\sqrt{-4}\times\sqrt{-9}=\sqrt{36}$. You must write $2i\times 3i=-6$ first.
- Giving $\arg(-1-i)=\frac{\pi}{4}$. The point is in the third quadrant, so the principal argument is $-\frac{3\pi}{4}$.
- Forgetting that $|z-a|=|z-b|$ is a line, not a circle.
- Dividing by $z-a$ without checking that a is the root you want.

Reading the question: what examiners expect

Further maths questions on complex numbers usually mix several ideas, so it helps to recognise the typical patterns. If the question says "find the roots" and gives you one complex root of a polynomial with real coefficients, you are expected to write down the conjugate immediately and build a real quadratic factor. If it says "express in the form $a+bi$", you must remove every i from the denominator and leave no brackets. If it says "show that", you must display every step, because the answer is already given and the marks are for the reasoning. When the question uses the words "exact value", keep surds and fractions of π in your answer instead of rounding. Finally, if a diagram is asked for, label the axes Re and Im, mark the key points with their coordinates, and shade the required region clearly.

Method: solving equations containing z and z^*

1. Write $z = x + iy$ and $z^* = x - iy$ at the start; do not try to solve for z directly.
2. Expand every bracket and collect the real parts and the imaginary parts separately.
3. Equate real parts and imaginary parts to form two simultaneous real equations in x and y.
4. Solve the pair, then write the answers back as complex numbers and substitute into the original equation as a check.

Worked example 11: an equation with a conjugate

Solve $z + 2z^* = 9 + 3i$.

Put $z = x + iy$, so that $z^* = x - iy$. Then the left-hand side is $x + iy + 2x - 2iy = 3x - iy$. Comparing real parts gives $3x = 9$, so $x = 3$. Comparing imaginary parts gives $-y = 3$, so $y = -3$. The solution is $z = 3 - 3i$. Substituting back, $z + 2z^* = 3 - 3i + 6 + 6i = 9 + 3i$, which is correct. The reason this method is reliable is that two complex numbers are equal exactly when both their real parts and their imaginary parts are equal, so every complex equation is really two real equations at once.

Key fact: the triangle inequality

For any two complex numbers, $|z_1 + z_2| \leq |z_1| + |z_2|$, with equality only when the two numbers have the same argument. Geometrically, one side of a triangle is never longer than the sum of the other two. This is the reason why, on the circle $|z - c| = r$, the greatest possible value of $|z|$ is $|c| + r$: the point farthest from the origin lies on the line from the origin through the centre.

Applications in science and engineering

Complex numbers are not just an exercise in algebra. In electrical engineering, resistors, coils and capacitors are combined by adding complex impedances, and the modulus and argument of the result give the size and timing of the current. In physics, the rotating arrow of a wave is the number $\cos \omega t + i \sin \omega t$, so adding two waves of the same frequency means adding two complex numbers. In computer graphics, rotating an image about the origin through an angle θ is done by multiplying every pixel position by the same complex number of modulus 1. Because multiplication combines a scaling and a rotation in a single operation, the algebra of complex numbers is a very economical way to describe plane geometry.

Method: revision plan for this chapter

1. Practise Cartesian arithmetic until you can divide complex numbers without hesitation.

2. Learn the four quadrant rules for the argument and always sketch the point.
3. Memorise the three standard loci and their Cartesian equations, and practise converting between the two descriptions.
4. Do at least three past-paper questions on polynomial roots, since these combine the conjugate pair theorem with algebraic division.
5. Finish by explaining each idea aloud in your own words; if you can teach it to a friend, you know it.

Understanding why the Argand diagram works

Every real number lives on a single number line, but a complex number needs two pieces of information, so it lives on a plane. The horizontal axis holds the real part and the vertical axis holds the imaginary part. Adding two complex numbers is exactly the same as adding two position vectors with the parallelogram rule, so the geometry you already know from vectors carries over unchanged. The new idea is multiplication. Because multiplying by i sends 1 to i, i to -1, -1 to $-i$ and $-i$ back to 1, it behaves like a quarter turn. Applying it twice gives a half turn, which is multiplication by $i^2 = -1$. This single observation explains why the square of the imaginary unit is negative: two quarter turns make a half turn, and a half turn takes every number to its negative. Once you see multiplication as a turn combined with a stretch, the rules for moduli and arguments become natural rather than something to memorise.

Worked example 12: finding a missing coefficient

The complex number $2 - 3i$ is a root of $z^2 + pz + q = 0$, where p and q are real. Find p and q in two different ways.

First method: because the coefficients are real, the other root is $2 + 3i$. The sum of the roots is 4 and equals $-p$, so $p = -4$. The product of the roots is $4 + 9 = 13$ and equals q. Second method: substitute $z = 2 - 3i$ directly. We have $(2 - 3i)^2 = 4 - 12i - 9 = -5 - 12i$, so $-5 - 12i + p(2 - 3i) + q = 0$. Equating the imaginary parts gives $-12 - 3p = 0$, so $p = -4$, and equating the real parts gives $-5 + 2p + q = 0$, so $q = 13$. Both methods agree, and it is good practice to confirm an answer by a second route whenever the time allows.

Watch out: sign errors with the argument

Most lost marks in this topic come from signs. When you divide two complex numbers in modulus-argument form you subtract the arguments, and the answer can fall outside the interval from $-\pi$ to π. If it does, add or subtract 2π to bring it back. For example, if $\arg z_1 = \frac{3\pi}{4}$ and $\arg z_2 = -\frac{\pi}{2}$, then $\arg(z_1 z_2^{-1}) = \frac{3\pi}{4} + \frac{\pi}{2} = \frac{5\pi}{4}$, which is not principal, so the answer is $\frac{5\pi}{4} - 2\pi = -\frac{3\pi}{4}$. Likewise, remember that the conjugate reverses the sign of the argument, and that $\arg(-z)$ differs from $\arg z$ by π, not by a sign change.

Worked example 13: loci with two conditions

Sketch the locus of z satisfying both $|z| = |z - 4|$ and $\arg(z - 1) = \frac{\pi}{4}$, and find the point of intersection.

The first condition is the perpendicular bisector of the points 0 and 4, which is the vertical line $x = 2$. The second is the half-line starting at 1 and rising at 45°, with equation $y = x - 1$ for $x > 1$. Substituting $x = 2$ gives $y = 1$, and since $2 > 1$ the point is on the half-line. The intersection is therefore the single point $z = 2 + i$. To check, compute $|2 + i| = \sqrt{5}$ and $|2 + i - 4| = |-2 + i| = \sqrt{5}$, and $\arg(1 + i) = \frac{\pi}{4}$. Both conditions hold.

Practical tips for the written paper

Work with exact values wherever you can, because exact answers are easier to check and are usually what the mark scheme asks for. When you solve a quadratic with a negative discriminant, write the square root of the negative number as an imaginary multiple straight away, for example $\sqrt{-36} = 6i$. Use the conjugate whenever you see a modulus squared. Keep your Argand sketches large and neat, label each axis, and draw dashed lines for strict inequalities. Lastly, if you are stuck on a locus question, write $z = x + iy$ and convert to Cartesian form; the algebra is often longer than the geometric argument, but it always works.

Connecting the topics in this chapter

It is worth stepping back to see how the sections of this chapter fit together, because examination questions deliberately cross the boundaries. Cartesian form is best when you add, subtract or divide, because the real and imaginary parts are visible. Modulus-argument form is best when you multiply, divide or take powers, because moduli multiply and arguments add. Loci link the algebra to geometry: a modulus condition measures distance, and an argument condition measures direction. Polynomial roots link everything to algebra, since a quadratic with a negative discriminant produces a conjugate pair, and the pair is symmetrical about the real axis on the Argand diagram. When you meet an unfamiliar problem, ask yourself which form makes the operation easiest, convert if necessary, and convert back at the end. Students who switch comfortably between the two forms tend to score highly in this topic, because they can choose the quickest route instead of forcing every question through the same method.

Worked example 14: a locus in context

A radio mast is at the point $4 + 3i$ on a map measured in kilometres, and its signal reaches 6 km. A walker is on the path described by $\arg(z + 2) = \frac{\pi}{2}$. Decide whether the walker is ever inside the signal range, and if so, for which section of the path.

The path is the half-line starting at the point -2 and going straight up, so the walker is

at $z = -2 + iy$ with $y > 0$. The distance from the mast is $\sqrt{(-2-4)^2 + (y-3)^2} = \sqrt{36 + (y-3)^2}$. The signal reaches 6 km, so we need $36 + (y-3)^2 \leq 36$, which forces $y = 3$. Therefore the walker is inside the range at exactly one point, $-2 + 3i$, where the path just touches the circle. This is a tangent situation, and it shows how a locus question can reduce to a simple condition once the distance formula is written down carefully.

Vocabulary to remember

The real part and the imaginary part of $x + iy$ are the numbers x and y; notice that the imaginary part does not include the letter i. A number whose real part is zero is called purely imaginary, and a number whose imaginary part is zero is real. The conjugate is the reflection in the real axis. The modulus is a distance, so it is never negative, and the argument is an angle measured anticlockwise from the positive real axis. The principal argument lies in the interval from $-\pi$ to π, excluding $-\pi$. A locus is a set of points, and a half-line is a locus with a starting point that is not included. Using these words precisely will help you write clear solutions and avoid ambiguity in explanations.

Worked example 15: using the conjugate to prove a result

Let $z = \dfrac{3+i}{1-2i}$. Find $|z|$ without finding z first, and then find z in the form $a + bi$.

Rather than calculate the quotient, use the fact that the modulus of a quotient is the quotient of the moduli. The numerator has modulus $\sqrt{9+1} = \sqrt{10}$ and the denominator has modulus $\sqrt{1+4} = \sqrt{5}$, so $|z| = \sqrt{10}/\sqrt{5} = \sqrt{2}$. For the form of z itself, multiply by the conjugate: $\dfrac{(3+i)(1+2i)}{5} = \dfrac{3+6i+i-2}{5} = \dfrac{1+7i}{5}$. The modulus of this is $\sqrt{1+49}/5 = \sqrt{50}/5 = \sqrt{2}$, which agrees. Notice that finding the modulus first is much quicker when only the size is required. This is a good habit: always ask what the question really needs before starting a long calculation.

Common misconceptions and how to avoid them

Many students believe that i is just another variable, but it is a fixed number defined by $i^2 = -1$, so any power of i can be reduced to one of four values. Others think that the argument of a negative real number is zero, when in fact it is π. Some try to order complex numbers using inequality signs, but there is no meaningful way to say that one complex number is greater than another; only moduli, which are real, can be compared. Another frequent error is to treat $|z|^2$ as z^2. They are different: z^2 is a complex number, whereas $|z|^2 = zz^*$ is real and non-negative. Keep these distinctions clear and you will avoid the majority of the errors seen in examination scripts.

Method: a checklist for any complex number question

1. Read the question carefully and decide which form of the complex number is the most convenient for the operation you have to perform, then write that form down.
2. Sketch a quick Argand diagram, even if the question does not ask for one, because the picture will reveal the quadrant, the symmetry and the geometry of the situation.
3. Carry out the algebra in small steps, replacing i^2 by -1 every time it appears, and keep the real and imaginary parts apart.
4. Decide whether the answer should be exact or rounded, and give the form the question asks for, such as $a + bi$ or modulus-argument form.
5. Check your final answer by substituting it back, comparing with your sketch, or repeating the calculation a different way. A sensible answer should have a modulus that matches the diagram and an argument in the correct quadrant.

How this chapter prepares you for the next

The ideas here are the foundation for the following chapter on de Moivre's theorem and roots of unity, where powers and roots of complex numbers are found by working in modulus-argument form. They also prepare you for polynomial equations, differential equations with oscillating solutions, and polar coordinates, where the pair of numbers modulus and argument becomes a new way of describing every point of the plane. Time spent making the basic skills automatic now will make those later topics much easier.

Worked example 16: from words to a locus

A robot vacuum cleaner must always stay twice as far from the charging dock at the origin as it is from the wall socket at the point 3. Describe the path it follows.

Translate the sentence into algebra first. Being twice as far from the dock as from the socket gives the relation that the distance to the dock equals twice the distance to the socket, that is $|z| = 2|z - 3|$. Now put $z = x + iy$ and square both sides to remove the square roots, giving $x^2 + y^2 = 4\left[(x - 3)^2 + y^2\right]$. Expanding the right-hand side and collecting terms leads to $3x^2 - 24x + 3y^2 + 36 = 0$, and dividing by 3 gives $x^2 - 8x + y^2 + 12 = 0$. Completing the square in x gives $(x - 4)^2 + y^2 = 4$. The robot therefore travels around a circle with centre at the point 4 and radius 2. Notice that this is not a perpendicular bisector: a perpendicular bisector appears only when the two distances are equal. When the ratio of the distances is any other positive constant, the locus is a circle, which was first described by the ancient Greek mathematician Apollonius.

Exam strategy and timing

On a typical further maths paper, a question on complex numbers carries between eight and fifteen marks and should take roughly one mark per minute. Spend the first minute reading and planning, because the most efficient method is not always the first one that comes to mind. Write each line of working clearly and underline or box the final answer. If you cannot complete one part, state the result you would expect and move on, since later parts often give you enough information to continue. Method marks are awarded for correct steps even when the final value is wrong, so never leave a blank. Leave a few minutes at the end to check signs, quadrant choices and the form in which the answer has been requested.

Did you know?

The number i was first used in the sixteenth century by Italian mathematicians such as Cardano and Bombelli, who found that solving certain cubic equations needed square roots of negative numbers, even when all three answers were real. Jean-Robert Argand published his diagram in 1806, and Gauss later used it widely. Today complex numbers are essential in electrical engineering, quantum physics, signal processing and the computer graphics used in films and games.

Otto's tip

Sketch the Argand diagram first. Most argument and locus questions become obvious from the picture, and the sketch also tells you the quadrant.