Worksheet - Complex numbers

Year: Further Maths · 20

Show your working.

Question 1

Question 1 – Arithmetic in Cartesian form *(Section A · Skills practice)*

Let $z = 3 - 2i$ and $w = 1 + 4i$. Write each answer in the form $a + bi$ and show your working.

a. $z + w$ and $2z - 3w$
b. zw
c. $\dfrac{z}{w}$, by multiplying top and bottom by the conjugate of w
d. zz^* and $|z|$

Question 2

Question 2 – Quadratic equations with complex roots *(Section A · Skills practice)*

Solve each quadratic equation, giving the roots in the form $p \pm qi$.

a. $z^2 - 6z + 25 = 0$
b. $z^2 + 2z + 10 = 0$
c. Find the quadratic equation with real coefficients that has $2 + i$ as a root.
d. For which real values of k does $z^2 + kz + 9 = 0$ have non-real roots?

Question 3

Question 3 – Modulus and argument *(Section A · Skills practice)*

Write each number in the form $r(\cos\theta + i\sin\theta)$ with $-\pi$. Sketch the point first to find the quadrant.

a. $1 + i$
b. $-\sqrt{3} + i$
c. $-2 - 2i$
d. $5 - 5\sqrt{3}\,i$

Question 4

Question 4 – Products and quotients *(Section A · Skills practice)*

Let $z_1 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ and $z_2 = 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$.

a. Find $|z_1 z_2|$ and $\arg(z_1 z_2)$, and hence write $z_1 z_2$ in the form $a + bi$.
b. Find $\left|\frac{z_1}{z_2}\right|$ and $\arg \frac{z_1}{z_2}$.
c. Find $|z_1^2|$ and $\arg(z_1^2)$.
d. Find $\arg(z_1 z_2^*)$.

Question 5

Question 5 – Sketching loci *(Section A · Skills practice)*

Describe each locus in words, give its Cartesian equation where possible, and sketch it.

a. $|z - 2| = 3$
b. $|z + 1 - i| \leq 2$
c. $\arg(z - 1) = \frac{\pi}{4}$
d. $|z - 1| = |z + 3|$

Question 6

Question 6 – Square roots of complex numbers *(Section A · Skills practice)*

Find the two square roots of each number in the form $a + bi$.

a. $8 - 6i$
b. $-15 + 8i$
c. Solve $z^2 = 3 + 4i$.

Question 7

Question 7 – Impedance in an a.c. circuit *(Section B · Real-life practice)*

In an alternating current circuit a resistor and an inductor in series have impedance $Z = 3 + 4i$ ohms: the real part is the resistance and the imaginary part is the reactance. A voltage $V = 20$ volts is applied, and the current is $I = V/Z$.

a. Find $|Z|$, and write the current I in the form $a + bi$.
b. Find $|I|$ amperes and the phase angle $\arg I$, correct to 3 decimal places. Does the current lead or lag the voltage?
c. A second branch has impedance $Z_2 = 6 - 8i$ and is connected in parallel with the first. The combined impedance satisfies $\frac{1}{Z_T} = \frac{1}{Z} + \frac{1}{Z_2}$. Find Z_T in the form $a + bi$, with a and b to 2 decimal places.

Question 8

Question 8 – The wind turbine *(Section B · Real-life practice)*

A wind turbine has three identical blades of length 45 m, spaced 120° apart. Model the

hub as the origin of an Argand diagram with distances in metres. At one instant the tip of blade A is at $z_A = 45\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$.

a. Write z_A in the form $a + bi$, giving exact values.
b. Explain why the tip of blade B is at $z_B = z_A\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ when it is the next blade anticlockwise. Find its modulus and argument.
c. Show that $z_A + z_B + z_C = 0$, where z_C is the third tip.
d. Find the exact distance between two blade tips.

Question 9

Question 9 – The drone and the charging point *(Section B · Real-life practice)*

A survey drone hovers at the point P with complex number $-3 + 2i$ (distances in metres). A charging point is at $C = 1 + i$. The drone is told to turn through 90° anticlockwise about C, which sends P to P' with $p' - c = i(p - c)$.

a. Find the complex number p'.
b. Find the distance PP'.
c. The drone then patrols the circle $|z - C| = 5$. Find the two points where the patrol circle crosses the real axis.

Question 10

Question 10 – A low-pass filter *(Section B · Real-life practice)*

An electronic filter has complex gain $G(\omega) = \dfrac{1}{1 + 0.01\,i\omega}$, where ω is the angular frequency of the signal in rad s^{-1}. The modulus $|G|$ tells you how much the signal is reduced, and $\arg G$ is the phase shift.

ω	100	300
G	?	?
$\lvert G\rvert$	?	?
$\arg G$	?	?

a. Complete the table, with G in the form $a + bi$ and $|G|$, $\arg G$ to 3 decimal places.
b. Show that $|G| = \dfrac{1}{\sqrt{1 + 10^{-4}\omega^2}}$ and explain why $|G|$ decreases as ω increases.
c. Find the frequency at which the signal is reduced to half its size.

Question 11

Question 11 – Navigating by two lighthouses *(Section B · Real-life practice)*

On a chart, lighthouse L_1 is at the origin and lighthouse L_2 is at the point $8 + 6i$ (distances in kilometres). A ship receives signals showing that it is the same distance

from both lighthouses. The ship is also within the 10 km range of L_1.

a. Write the ship's condition as $|z| = |z - 8 - 6i|$ and show that its Cartesian equation is $4x + 3y = 25$.
b. Write down the inequality for the range of L_1 and find where the line $4x + 3y = 25$ meets the circle $|z| = 10$. Give exact coordinates.
c. Hence find the length of ship track that lies inside the range of L_1.

Question 12

Question 12 – Fractal images on a computer *(Section B · Real-life practice)*

Computer graphics programs draw fractals by repeating $z_{n+1} = z_n^2 + c$ from $z_0 = 0$, where c is a complex number. If $|z_n|$ ever exceeds 2, the sequence escapes to infinity and the point c is coloured as outside the set.

a. Take $c = i$. Find z_1, z_2, z_3, z_4.
b. What happens next? Explain why $c = i$ is inside the set.
c. Take $c = 1 + i$ and find z_1, z_2, z_3. Show that $|z_3| > 2$.

Question 13

Question 13 – Is the bridge stable? *(Section B · Real-life practice)*

Engineers testing the vibrations of a footbridge find that its response depends on the roots of the characteristic cubic $z^3 + 6z^2 + 21z + 26 = 0$. The bridge is stable only when every root has a negative real part.

a. Show that $z = -2$ is a root.
b. Find the other two roots in the form $p \pm qi$.
c. Is the bridge stable? Explain how the roots show it.
d. Sketch the three roots on an Argand diagram and state the sum of the roots.

Question 14

Question 14 – Combining two tides *(Section B · Real-life practice)*

Two tidal effects at a harbour have heights (in metres) $h_1 = 3\cos\omega t$ and $h_2 = 2\cos\left(\omega t + \frac{\pi}{3}\right)$. Oceanographers add them by treating each as the real part of a complex number, $3e^{i\omega t}$ and $2e^{i(\omega t + \pi/3)}$.

a. Write the sum of the amplitudes $3 + 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ in the form $a + bi$.
b. Find the modulus and argument of this number to 3 decimal places.
c. Hence write $h_1 + h_2$ in the form $R\cos(\omega t + \phi)$, and state the greatest height.

Question 15

Question 15 – A quartic with four complex roots *(Section C · Problem solving)*

The quartic polynomial $f(z) = z^4 - 6z^3 + 18z^2 - 30z + 25$ has real coefficients.

a. Show that $z = 2 + i$ is a root of $f(z) = 0$.
b. Write down a second root, and hence find a quadratic factor of $f(z)$ with real coefficients.
c. Factorise $f(z)$ completely over the real numbers and solve $f(z) = 0$.
d. Plot the four roots on an Argand diagram. What shape do they make and what is its area?

Question 16

Question 16 – Proving the parallelogram law *(Section C · Problem solving)*

Let z_1 and z_2 be any complex numbers.

a. Use $|z|^2 = zz^*$ to prove that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$.
b. Check the result for $z_1 = 3 + i$ and $z_2 = 1 - 2i$.
c. Interpret the result geometrically for the parallelogram with vertices $0, z_1, z_1 + z_2, z_2$.
d. Prove that $z + \dfrac{1}{z}$ is real if and only if z is real or $|z| = 1$.

Question 17

Question 17 – A region of the Argand diagram *(Section C · Problem solving)*

The region R is defined by $|z - 3 - 4i| \leq 5$.

a. Show that the origin lies on the boundary of R and find the Cartesian equation of the boundary.
b. Find where the boundary crosses the imaginary axis.
c. Find the greatest and least values of $|z|$ for points in R, and the complex number at which the greatest value occurs.
d. The tangent to the circle at the origin has gradient $-\frac{3}{4}$. Find the greatest and least values of $\arg z$ for points on the boundary other than the origin, to 3 decimal places.
e. Find the area of the part of R with $\mathrm{Re}(z) \geq 3$.

Question 18

Question 18 – An equilateral formation *(Section C · Problem solving)*

Three drones fly in an equilateral triangle formation. In an Argand diagram representing the sky, the drones are at the points 0, z_1 and z_2.

a. Suppose $z_2 = z_1\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$. Explain why the triangle is equilateral.
b. Show that $z_1^2 + z_2^2 = z_1 z_2$.
c. Take $z_1 = 6$. Find z_2 in the form $a + bi$ and verify the identity numerically.

d. Find the position of the centroid $\frac{1}{3}(0 + z_1 + z_2)$ and show that it is the same distance from all three drones.

Question 19

Question 19 – Finding the coefficients *(Section C · Problem solving)*

The cubic equation $z^3 + pz^2 + qz + r = 0$ has real coefficients p, q, r. Two of its roots are $3 + 2i$ and $3 - 2i$, and the third root is the real number α. The product of all three roots is 13.

a. Explain why $3 - 2i$ must be a root.
b. Find α.
c. Find p, q and r.
d. Verify your cubic by substituting $z = 3 + 2i$.
e. Write the cubic as the product of a linear and a real quadratic factor.

Question 20

Question 20 – Challenge: the imaginary quotient *(Section D · Challenge)*

Let $z = \cos\theta + i\sin\theta$ with $-\pi$, so that $|z| = 1$.

a. Show that $1 + z = 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$ and find $|1 + z|$ and $\arg(1 + z)$.
b. Hence show that $\dfrac{z - 1}{z + 1}$ is purely imaginary, and find it in terms of θ.
c. Check your answer for $z = i$.
d. Explain geometrically why the points 0, -1 and 1 give a right angle at z when z is on the unit circle.