Answers - Complex numbers

Year: Further Maths

Answer to Question 1

Method: treat i like a letter and replace i^2 by -1. To divide, multiply the top and bottom by the conjugate of the denominator, so that the denominator becomes the real number $a^2 + b^2$.

a. Add or subtract the real parts and the imaginary parts separately. $z + w = 4 + 2i$. $2z - 3w = (6 - 4i) - (3 + 12i) = 3 - 16i$ So $z + w = 4 + 2i$ and $2z - 3w = 3 - 16i$.

b. Expand the brackets and use $i^2 = -1$, so $-8i^2 = +8$.
$zw = 3 + 12i - 2i - 8i^2 = 3 + 10i + 8 = 11 + 10i$

c. The conjugate of w is $1 - 4i$ and $(1 + 4i)(1 - 4i) = 1 + 16 = 17$.
$\frac{(3 - 2i)(1 - 4i)}{(1 + 4i)(1 - 4i)} = \frac{3 - 12i - 2i + 8i^2}{17} = \frac{-5 - 14i}{17} = -\frac{5}{17} - \frac{14}{17}i$ Check:
$w \times \left(-\frac{5}{17} - \frac{14}{17}i\right) = \frac{-5-14i-20i+56}{17} = 3 - 2i$.

d. The product of a number and its conjugate is always real: $zz^* = |z|^2$.
$zz^* = (3 - 2i)(3 + 2i) = 9 + 4 = \mathbf{13}$, so $|z| = \sqrt{13}$

Answer to Question 2

Method: a polynomial with real coefficients has its non-real roots in conjugate pairs $p \pm qi$. For a quadratic, use the formula with $\sqrt{-k} = i\sqrt{k}$, or use the sum and product of the roots.

a. Use the quadratic formula; the discriminant is negative.
$z = \frac{6 \pm \sqrt{36 - 100}}{2} = \frac{6 \pm 8i}{2} = 3 \pm 4i$ The roots form a conjugate pair.

b. The discriminant is $4 - 40 = -36$, so $\sqrt{-36} = 6i$.
$z = \frac{-2 \pm \sqrt{4 - 40}}{2} = \frac{-2 \pm 6i}{2} = -1 \pm 3i$ Check: sum -2 and product $1 + 9 = 10$.

c. For real coefficients the conjugate is also a root; then use $z^2 - (\text{sum})z + \text{product} = 0$. The other root is $2 - i$. Sum $= 4$, product $= 4 + 1 = 5$, so $z^2 - 4z + 5 = 0$. Check: $(2 + i)^2 - 4(2 + i) + 5 = 3 + 4i - 8 - 4i + 5 = 0$.

d. The roots are non-real when the discriminant is negative. Need $k^2 - 36$, so -6.

Answer to Question 3

Method: for $z = x + iy$, $|z| = \sqrt{x^2 + y^2}$. Plot the point first to find the quadrant, then find the reference angle and adjust it so that $-\pi$.

a. $r = \sqrt{2}$, first quadrant, $\theta = \frac{\pi}{4}$: $\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$
b. $r = 2$, second quadrant, reference angle $\frac{\pi}{6}$, $\theta = \frac{5\pi}{6}$: $2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$
c. $r = 2\sqrt{2}$, third quadrant, reference angle $\frac{\pi}{4}$, $\theta = -\frac{3\pi}{4}$: $2\sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right)$
d. $r = \sqrt{25+75} = 10$, fourth quadrant, $\theta = -\frac{\pi}{3}$: $10\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$

Answer to Question 4

Method: in modulus-argument form, multiplying multiplies the moduli and adds the arguments; dividing divides the moduli and subtracts the arguments. The conjugate has the same modulus and the opposite argument.

a. Multiply moduli, add arguments: $|z_1 z_2| = 8$, $\arg = \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}$. So $z_1 z_2 = 8(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) = 8i$
b. Divide moduli, subtract arguments: modulus $\frac{1}{2}$, argument $\frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$
c. $|z_1|^2 = 4$ and $\arg = 2 \times \frac{\pi}{3} = \frac{2\pi}{3}$
d. $\arg z_2^* = -\frac{\pi}{6}$, so $\arg(z_1 z_2^*) = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$

So working in modulus-argument form avoids expanding brackets altogether.

Answer to Question 5

a. A circle, centre $(2, 0)$, radius 3: $(x-2)^2 + y^2 = 9$.
b. The boundary and inside of a circle, centre $(-1, 1)$, radius 2: $(x+1)^2 + (y-1)^2 \leq 4$.
c. A half-line starting at the point $(1, 0)$ (not included) at 45° above the real direction: $y = x - 1,\ x > 1$.
d. Perpendicular bisector of the points 1 and -3: $(x-1)^2 + y^2 = (x+3)^2 + y^2 \Rightarrow -2x + 1 = 6x + 9 \Rightarrow x = -1$.

Answer to Question 6

Method: to find $\sqrt{a+bi}$, write $(x+iy)^2 = a + bi$ and compare real and imaginary parts: $x^2 - y^2 = a$ and $2xy = b$. Eliminate y to get a quadratic in x^2; x must be real, so reject the negative value of x^2.

a. Let $(x+iy)^2 = 8 - 6i$: $x^2 - y^2 = 8$ and $2xy = -6$. Then $y = -3/x$ and $x^4 - 8x^2 - 9 = 0$, so $(x^2-9)(x^2+1) = 0$, $x = \pm 3$. Roots: $\pm(3 - i)$. Check: $(3-i)^2 = 9 - 6i - 1 = 8 - 6i$.
b. $x^2 - y^2 = -15$, $xy = 4$: $x^4 + 15x^2 - 16 = 0$, $(x^2+16)(x^2-1) = 0$, $x = \pm 1$, $y = \pm 4$. Roots: $\pm(1 + 4i)$. Check: $1 - 16 + 8i$.
c. $x^2 - y^2 = 3$, $xy = 2$: $x^4 - 3x^2 - 4 = 0$, $(x^2-4)(x^2+1) = 0$, so $z = \pm(2 + i)$.

Answer to Question 7

Method: treat i like a letter and replace i^2 by -1. To divide, multiply the top and bottom by the conjugate of the denominator, so that the denominator becomes the real number $a^2 + b^2$.

a. $|Z| = \sqrt{9 + 16} =$ **5 ohms**. $I = \dfrac{20(3 - 4i)}{25} = 2.4 - 3.2i$

b. $|I| = \sqrt{5.76 + 10.24} =$ **4 A**. $\arg I = -\arctan \frac{4}{3} = -0.927$ **rad**. The argument is negative, so the current **lags** the voltage.

c. $\dfrac{1}{Z} = \dfrac{3 - 4i}{25}$ and $\dfrac{1}{Z_2} = \dfrac{6 + 8i}{100}$. Sum $= \dfrac{12 - 16i + 6 + 8i}{100} = \dfrac{18 - 8i}{100}$. So $Z_T = \dfrac{100}{18 - 8i} = \dfrac{100(18 + 8i)}{388} = \dfrac{450 + 200i}{97} \approx 4.64 + 2.06i$ **ohms**.

Answer to Question 8

a. $45 \cos \frac{\pi}{6} = \frac{45\sqrt{3}}{2}$ and $45 \sin \frac{\pi}{6} = \frac{45}{2}$: $z_A = \dfrac{45\sqrt{3}}{2} + \dfrac{45}{2}i$

b. Multiplying by a number of modulus 1 and argument $\frac{2\pi}{3}$ rotates z_A anticlockwise through 120° about the origin. $|z_B| =$ **45** and $\arg z_B = \frac{\pi}{6} + \frac{2\pi}{3} = \frac{5\pi}{6}$.

c. The tips are at arguments $\frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{3\pi}{2}$.
$z_A + z_B = 45(\cos \frac{\pi}{6} + \cos \frac{5\pi}{6}) + 45i(\sin \frac{\pi}{6} + \sin \frac{5\pi}{6}) = 0 + 45i = 45i$ and $z_C = 45(0 - i) = -45i$. The sum is **0**.

d. The tips form an equilateral triangle inscribed in a circle of radius 45, so the side is $45\sqrt{3}$. Check: $|z_A - z_B| = |45\sqrt{3}| = 45\sqrt{3} \approx 77.9$ **m**.

Answer to Question 9

Method: multiplying by i rotates a vector through 90° anticlockwise. To rotate about C instead of the origin, rotate the vector $p - c$, then add c back.

a. $p - c = -3 + 2i - 1 - i = -4 + i$. $i(-4 + i) = -4i - 1$. So $p' = 1 + i - 1 - 4i = -3i$. So the drone moves to $-3i$, that is the point $(0, -3)$.

b. The distance between two points is the modulus of the difference.
$p' - p = -3i - (-3 + 2i) = 3 - 5i$, so $PP' = \sqrt{9 + 25} = \sqrt{34} \approx 5.83$ **m**. Check: the rotation keeps $|p - c| = \sqrt{17}$, and $PP' = \sqrt{2} \times \sqrt{17} = \sqrt{34}$.

c. Points on the real axis have the form $x + 0i$; substitute into the equation of the circle. $|x + 0i - 1 - i| = 5$ gives $(x - 1)^2 + 1 = 25$, $(x - 1)^2 = 24$, $x = 1 \pm 2\sqrt{6}$ (about 5.90 and -3.90). So the circle crosses the real axis at $1 + 2\sqrt{6}$ and $1 - 2\sqrt{6}$.

Answer to Question 10

Method: treat i like a letter and replace i^2 by -1. To divide, multiply the top and bottom by the conjugate of the denominator, so that the denominator becomes the real number $a^2 + b^2$.

a. $G(100) = \dfrac{1}{1+i} = \dfrac{1-i}{2}$, $|G| = \dfrac{1}{\sqrt{2}} \approx 0.707$, $\arg G = -\frac{\pi}{4} \approx -0.785$.
$G(300) = \dfrac{1}{1+3i} = \dfrac{1-3i}{10}$, $|G| = \dfrac{1}{\sqrt{10}} \approx 0.316$, $\arg G = -\arctan 3 \approx -1.249$.

b. $|1 + 0.01i\omega| = \sqrt{1 + 10^{-4}\omega^2}$, and $|1/a| = 1/|a|$. The denominator grows with ω, so $|G|$ falls: the filter weakens high frequencies.

c. Need $\sqrt{1 + 10^{-4}\omega^2} = 2$, so $10^{-4}\omega^2 = 3$, $\omega = 100\sqrt{3} \approx$ **173 rad s^{-1}**.

Answer to Question 11

a. With $z = x + iy$: $x^2 + y^2 = (x-8)^2 + (y-6)^2$. Expanding, $0 = -16x + 64 - 12y + 36$, so $4x + 3y = 25$.

b. Range: $|z| \le 10$. Substituting $y = \dfrac{25 - 4x}{3}$: $9x^2 + (25-4x)^2 = 900$, $25x^2 - 200x - 275 = 0$, $x^2 - 8x - 11 = 0$, $x = 4 \pm 3\sqrt{3}$. Then $y = 3 \mp 4\sqrt{3}$. Points $(4 + 3\sqrt{3},\ 3 - 4\sqrt{3})$ and $(4 - 3\sqrt{3},\ 3 + 4\sqrt{3})$.

c. The distance between them is $\sqrt{(6\sqrt{3})^2 + (8\sqrt{3})^2} = \sqrt{108 + 192} = \sqrt{300} = 10\sqrt{3} \approx 17.3$ **km**. (The line is 5 km from L_1, so the chord is $2\sqrt{100 - 25}$.)

Answer to Question 12

Method: treat i like a letter and replace i^2 by -1. To divide, multiply the top and bottom by the conjugate of the denominator, so that the denominator becomes the real number $a^2 + b^2$.

a. $z_1 = i$. $z_2 = i^2 + i = -1 + i$. $z_3 = (-1+i)^2 + i = -2i + i = -i$.
$z_4 = (-i)^2 + i = -1 + i$.

b. $z_4 = z_2$, so the values repeat $-1+i, -i, -1+i, -i, \ldots$ for ever. Both have modulus at most $\sqrt{2}$, so the sequence never escapes: $c = i$ is inside.

c. $z_1 = 1 + i$. $z_2 = (1+i)^2 + 1 + i = 2i + 1 + i = 1 + 3i$.
$z_3 = (1+3i)^2 + 1 + i = 1 + 6i - 9 + 1 + i = -7 + 7i$. $|z_3| = 7\sqrt{2} \approx 9.9 > 2$, so $c = 1 + i$ escapes.

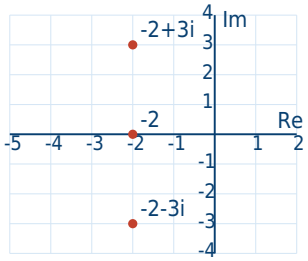
Answer to Question 13

a. $(-2)^3 + 6(4) + 21(-2) + 26 = -8 + 24 - 42 + 26 = \mathbf{0}$.

b. Divide by $(z+2)$: $z^3 + 6z^2 + 21z + 26 = (z+2)(z^2 + 4z + 13)$. Then $z = \dfrac{-4 \pm \sqrt{16 - 52}}{2} = -2 \pm 3i$.

c. The real parts are $-2, -2, -2$, all negative, so **the bridge is stable**.

d. Points $(-2, 0)$, $(-2, 3)$ and $(-2, -3)$ lie on a vertical line. Their sum is -6, which equals $-b/a$ for the cubic.



Answer to Question 14

Method: for $z = x + iy$, $|z| = \sqrt{x^2 + y^2}$. Plot the point first to find the quadrant, then find the reference angle and adjust it so that $-\pi$.

a. $\cos\frac{\pi}{3} = \frac{1}{2}$ and $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$: $3 + 1 + \sqrt{3}\,i = 4 + \sqrt{3}\,i$.

b. $R = \sqrt{16 + 3} = \sqrt{19} \approx \mathbf{4.359}$; $\phi = \arctan\dfrac{\sqrt{3}}{4} \approx \mathbf{0.409\ rad}$.

c. $h_1 + h_2 = \sqrt{19}\cos(\omega t + 0.409)$. The greatest height is $\sqrt{19} \approx \mathbf{4.36\ m}$.

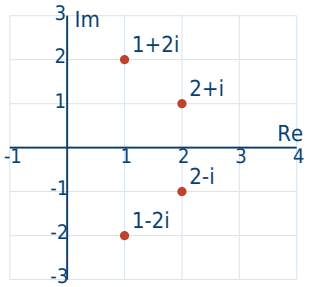
Answer to Question 15

a. $(2+i)^2 = 3 + 4i$. $(2+i)^3 = (3+4i)(2+i) = 2 + 11i$.
$(2+i)^4 = (3+4i)^2 = -7 + 24i$.
$f = -7 + 24i - 6(2 + 11i) + 18(3 + 4i) - 30(2 + i) + 25 = (-7 - 12 + 54 - 60 + 25) + (24 - 66 + 72 - 30)i = 0 + 0i =$ **0**.

b. The conjugate $2 - i$ is also a root. $(z - 2 - i)(z - 2 + i) = (z-2)^2 + 1 = z^2 - 4z + 5$.

c. Divide: $f(z) = (z^2 - 4z + 5)(z^2 - 2z + 5)$. The second factor has roots $\dfrac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i$. So $z = 2 \pm i,\ 1 \pm 2i$.

d. The points $(2, \pm 1)$ and $(1, \pm 2)$ form an isosceles trapezium with parallel sides of length 2 (at $x = 2$) and 4 (at $x = 1$) a distance 1 apart. Area $= \frac{1}{2}(2 + 4) \times 1 = \mathbf{3}$.



Answer to Question 16

a. $|z_1+z_2|^2=(z_1+z_2)(z_1^*+z_2^*)=|z_1|^2+z_1z_2^*+z_2z_1^*+|z_2|^2$. Similarly $|z_1-z_2|^2=|z_1|^2-z_1z_2^*-z_2z_1^*+|z_2|^2$. Adding, the cross terms cancel, giving $2(|z_1|^2+|z_2|^2)$.

b. $|4-i|^2=17$, $|2+3i|^2=13$, total 30. $|z_1|^2=10$, $|z_2|^2=5$, so $2(10+5)=$ **30**. They agree.

c. The diagonals are z_1+z_2 and z_1-z_2. The sum of the squares of the diagonals equals the sum of the squares of all four sides, since the sides have lengths $|z_1|, |z_2|, |z_1|, |z_2|$.

d. Put $z=x+iy$. $z+\dfrac{1}{z}=x+iy+\dfrac{x-iy}{x^2+y^2}$, with imaginary part $y\left(1-\dfrac{1}{x^2+y^2}\right)$. It is zero when $y=0$ (real z) or $x^2+y^2=1$ (that is $|z|=1$). **Both cases are the only ones.**

Answer to Question 17

a. $|0-3-4i|=\sqrt{9+16}=5$, so 0 is on the boundary. Equation: $(x-3)^2+(y-4)^2=25$.

b. Put $x=0$: $9+(y-4)^2=25$, $(y-4)^2=16$, $y=0$ or 8. The points are 0 **and** $8i$.

c. The centre is at distance 5 from the origin and the radius is 5. Greatest $|z|=5+5=$ **10**, at $z=2(3+4i)=6+8i$. Least $|z|=$ **0**, at the origin.

d. The tangent at the origin is perpendicular to $3+4i$, so it has direction $-4+3i$ (argument $\pi-\arctan\frac{3}{4}$) or $4-3i$ (argument $-\arctan\frac{3}{4}$). The greatest $\arg z$ approaches **2.498** and the least approaches -0.644 (neither is reached exactly).

e. The line $x=3$ passes through the centre, so $\frac{1}{2}$ of the disc is taken: $\frac{25\pi}{2}\approx 39.3$.

Answer to Question 18

a. The side from 0 to z_2 has the same length as the side from 0 to z_1 (the modulus is unchanged) and the angle between them is 60°. An isosceles triangle with a 60° apex angle is equilateral.

b. Let $\omega=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}=\frac{1}{2}+\frac{\sqrt{3}}{2}i$. Then $\omega^2=-\frac{1}{2}+\frac{\sqrt{3}}{2}i=\omega-1$. So

$z_1^2 + z_2^2 = z_1^2(1+\omega^2) = z_1^2\omega = z_1 z_2$. Hence $z_1^2 + z_2^2 = z_1 z_2$.

c. $z_2 = 6\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 3 + 3\sqrt{3}\,i$. $z_1^2 = 36$, $z_2^2 = 9 - 27 + 18\sqrt{3}\,i = -18 + 18\sqrt{3}\,i$, $z_1 z_2 = 18 + 18\sqrt{3}\,i$. $36 - 18 + 18\sqrt{3}\,i = 18 + 18\sqrt{3}\,i$. Verified.

d. Centroid $= \frac{1}{3}(9 + 3\sqrt{3}\,i) = 3 + \sqrt{3}\,i$, modulus $\sqrt{9+3} = 2\sqrt{3}$. Distance to z_1: $|3 - \sqrt{3}\,i| = 2\sqrt{3}$. Distance to z_2: $|2\sqrt{3}\,i| = 2\sqrt{3}$. **All equal $2\sqrt{3}$.**

Answer to Question 19

a. The coefficients are real, so complex roots occur in conjugate pairs (conjugate root theorem).

b. Product of roots $= (3+2i)(3-2i)\alpha = 13\alpha = 13$, so $\alpha = 1$.

c. Sum of roots $= 7 = -p$, so $p = -7$. Sum of pairs $= 13 + 6\alpha = 19 = q$. Product $= 13 = -r$, so $r = -13$. The cubic is $z^3 - 7z^2 + 19z - 13 = 0$.

d. $(3+2i)^2 = 5 + 12i$, $(3+2i)^3 = (5+12i)(3+2i) = -9 + 46i$. $-9 + 46i - 7(5+12i) + 19(3+2i) - 13 = (-9 - 35 + 57 - 13) + (46 - 84 + 38)i = 0$. Verified.

e. $(z-1)(z^2 - 6z + 13)$.

Answer to Question 20

a. $1 + \cos\theta = 2\cos^2\frac{\theta}{2}$ and $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$. So $1 + z = 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$. For $-\pi$, $\cos\frac{\theta}{2} > 0$, so $|1+z| = 2\cos\frac{\theta}{2}$ and $\arg(1+z) = \frac{\theta}{2}$.

b. Similarly $z - 1 = -2\sin^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2i\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$. Dividing, $\dfrac{z-1}{z+1} = i\tan\frac{\theta}{2}$, purely imaginary.

c. $\theta = \frac{\pi}{2}$: $\dfrac{i-1}{i+1} = \dfrac{(i-1)(1-i)}{2} = \dfrac{i - i^2 - 1 + i}{2} = \dfrac{2i}{2} = i$, and $i\tan\frac{\pi}{4} = i$. Correct.

d. $\arg\dfrac{z-1}{z+1} = \pm\frac{\pi}{2}$ is the angle between the lines from z to 1 and from z to -1. So the angle subtended by the diameter is 90°, which is the angle in a semicircle theorem.